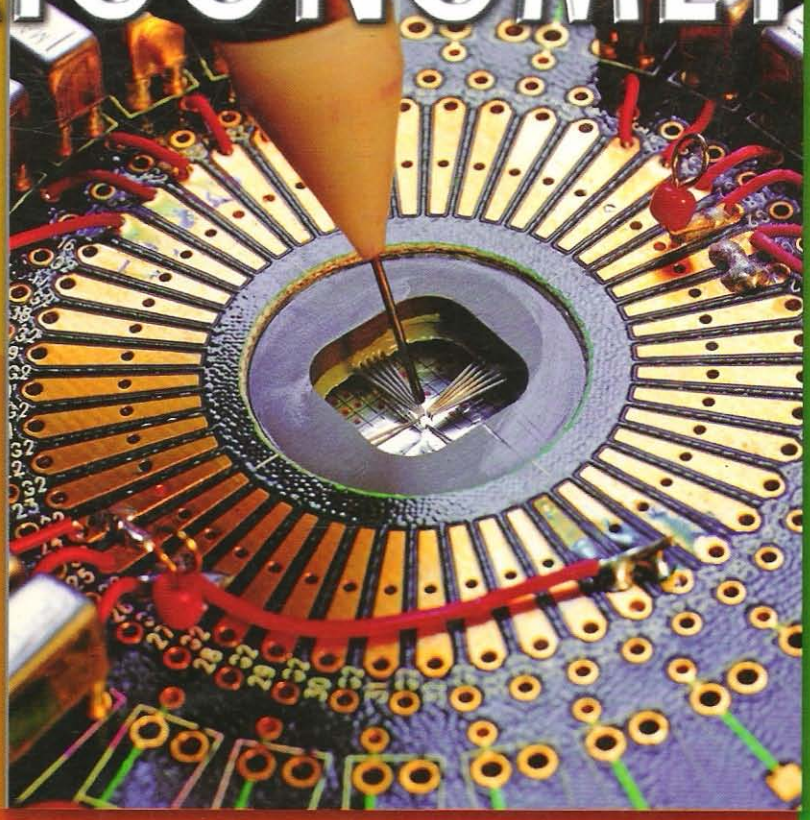
Introduction to TRIGONOMETRY



MODULAR SYSTEM

Zambak PUBLISHING

MATHEMATICS SERIES

MODULAR SYSTEM

Introduction to TRIGONOMETRY

İlker Tantürk

Zambak PUBLISHING

http://book.zambak.com

Zambak
PUBLISHING

Copyright © Zambak Basım Yayın Eğitim ve Turizm İşletmeleri Sanayi Ticaret A.Ş.

All rights reserved.
No part of this book may be reproduced, stored in a retrieval system or transmitted in any form without the prior written permission of the publisher.

Digital Assembly
Zambak Typesetting & Design

Page Design
Şamil Keskinoğlu

Proofreader
Zoe Barnett

Publisher
Zambak Basım Yayın Eğitim ve Turizm İşletmeleri Sanayi Ticaret A.Ş.

Printed by
Çağlayan A.Ş. Sarnıç Yolu Üzeri No:7
Gaziemir / Izmir, August 2007
Tel: 0-232-252 22 85
0-232-522-20-96-97

ISBN: 975-266-471-7
Printed in Turkey

DISTRIBUTION
ZAMBAK YAYINLARI
Bulgurlu Mah. Haminne Çeşmesi Sok.
No. 20 34696 Üsküdar / Istanbul

Tel.: +90-216 522 09 00 (pbx)
Fax: +90-216 443 98 39
http://book.zambak.com



PREFACE

To the Teacher

This book is an introduction to trigonometry and trigonometric applications for high school students. Before beginning this book, students should be familiar with the basic properties of of angles, triangles and circles, and be able to manipulate and factorize algebraic expressions. The book is divided into five sections, structured as follows:

- *The first section covers the concepts of angle, arc and circle and their basic properties. It also introduces the degree and radian units of angle measure.*
- *The second section is an introduction to right triangle trigonometry, and introduces key trigonometric identities which are used throughout the rest of the book.*
- *The third section deals with trigonometry on the unit circle, and leads students to an understanding of the six main trigonometric functions and their properties.*
- *The fourth section builds on the material of the previous sections and presents the main trigonometric theorems and formulas that students need to know at this level, with many examples of how they are used.*
- *The fifth section concludes the book with a look at some applied trigonometry problems, bringing together the material that student have studied during the course.*

This book has been designed to be an effective teaching aid, and includes all of the features of the Zambak Modular System high school math teaching series:

- *The book uses a linear teaching approach, with material in the latter sections building on concepts and math covered previously in the text.*
- *Self-test 'Check Yourself' sections at key points in the text allow students to check their understanding, and build students' confidence for further study.*
- *Exercises at the end of each section give students a chance to use the skills and techniques they have learned to solve both pure and applied problems.*
- *A chapter summary at the end of the book provides a concise review of the main content of the chapter. Included in the summary are a set of concept check questions which ask students to explain key concepts and ideas in their own words.*
- *Review tests at the end of the book cover the material in the whole book, and help to prepare students for exams.*

Acknowledgements

Many friends and colleagues were of great help during the writing of this book. I would like to thank everybody who helped me at Zambak publications, especially Mustafa Kırıkçı and Cem Giray. Special thanks also go to Şamil Keskinoğlu for his patient typesetting and design.

İlker Tantürk

To the Student: Using This Book

This book is designed so that you can use it effectively. Each section has its own special color that you can see at the bottom of the page.

Different pieces of information in this book are useful in different ways. Look at the types of information, and how they appear in the book:

Section 1 — *Fundamentals of Trigonometry*

Section 2 — *Right Triangle Trigonometry*

Section 3 — *Trigonometric Functions of Real Numbers*

Section 4 — *Trigonometric Theorems and Formulas*

Section 5 — *Applied Trigonometry*

Note

The **rad** unit or exponential ^{R} notation
radians. If no unit is specified fo
radians. For

Notes help you focus on important details. When you see a note, read it twice! Make sure you understand it.

Definition grad

When the circu
correspondi

Theorem law of sines

Let ABC be a triangl
which is inscri

Remark 1

On a unit circle,
angle in radians i
of the

Definition boxes give formal descriptions of new concepts. Theorem boxes include propositions that can be proved. The information in these boxes is very important for further understanding and for solving examples.

EXAMPLE 9 The circumferenc
corresponding to

Solution The compl

Examples include problems related to the topic and their solution, with explanations. The examples are numbered, so you can find them easily in the book.

Check Yourself 5

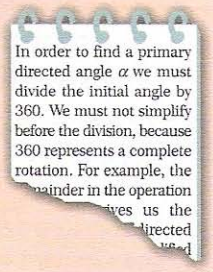
1. In a right triangle, θ is an acute angl
cotangent ratios of the same angle.
2. One leg of an isosceles right triangle i
of the two equal acute angles

Answers

1. $\sin\theta = \frac{4}{}$

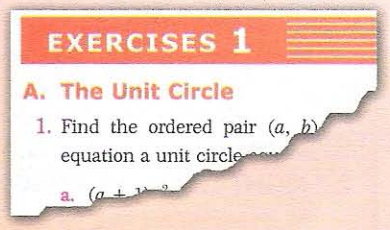
Check Yourself sections help you check your understanding of what you have just studied. Solve Check Yourself questions alone and then compare your answers with the answer key provided. If your answers are correct, you can move on to the next section. If your answer is wrong, go through your working again and check back through the examples in the section.

A small notebook in the left or right margin of a page reminds you of material that is related to the topic you are studying. It might help you to see your mistakes, too! Notebooks are the same color as the section you are studying.


In order to find a primary directed angle α we must divide the initial angle by 360. We must not simplify before the division, because 360 represents a complete rotation. For example, the ...ainder in the operation ...ives us the ...directed ...

094

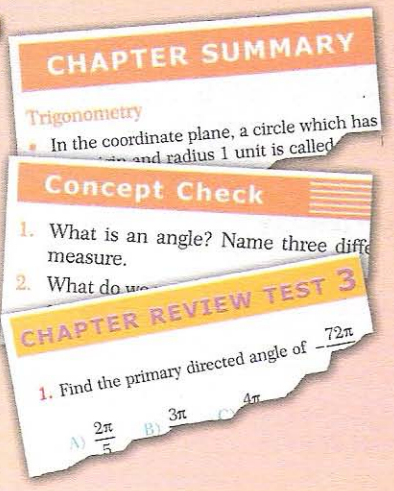
Exercises at the end of each section cover the material in the whole section. You should be able to solve all the problems which do not have a star. One star. (✪) next to a question means the question is a bit more difficult. Two stars (✪✪) next to a question mean the question is for students who are looking for a challenge! The answers to the exercises are at the back of the book.


EXERCISES 1

A. The Unit Circle

1. Find the ordered pair (a, b) ... equation a unit circle ...

a. $(a + 1)^2 + \ldots$


CHAPTER SUMMARY

Trigonometry

• In the coordinate plane, a circle which has ... and radius 1 unit is called ...

Concept Check

1. What is an angle? Name three diff... measure.
2. What do w...

CHAPTER REVIEW TEST 3

1. Find the primary directed angle of $-\frac{72\pi}{\ldots}$

A) $\frac{2\pi}{5}$ B) $\frac{3\pi}{\ldots}$ C) $\frac{4\pi}{\ldots}$

The Chapter Summary at the end of the book summarizes all the important material that has been covered in the book. The Concept Check section contains oral questions. In order to answer them you don't need a paper or pen. If you answer Concept Check questions correctly, it means you know that topic! The answers to these questions are in the material you studied. Go back over the material if you are not sure about an answer to a Concept Check question. After the Concept Check there are Chapter Review Tests in increasing order of difficulty with multiple choice questions. The answer key for these tests is at the back of the book.

CONTENTS

Introduction to TRIGONOMETRY

1. FUNDAMENTALS OF TRIGONOMETRY 10
 - A. THE UNIT CIRCLE 10
 - B. ANGLES AND DIRECTION 12
 - 1. The Concept of Angle 12
 - 2. Directed Angles 12
 - 3. Directed Arcs 13
 - C. UNITS OF ANGLE MEASURE 14
 - 1. Grad 14
 - 2. Degree 15
 - 3. Radian 17
 - 4. Converting Units of Angle Measure 19
 - D. PRIMARY DIRECTED ANGLES 21
 - 1. Coterminal Angles 21
 - 2. Primary Directed Angles and Arcs 23

 EXERCISES 1 27

2. RIGHT TRIANGLE TRIGONOMETRY ...29
 - A. TRIGONOMETRIC RATIOS 29
 - 1. Definition 29
 - 2. Special Triangles and Ratios 32
 - B. TRIGONOMETRIC IDENTITIES 34
 - 1. Basic Identities 34
 - a. Pythagorean identities 34
 - b. Tangent and cotangent identities ..35
 - c. Reciprocal identities 36
 - 2. Simplifying Trigonometric Expressions 38
 - 3. Verifying Trigonometric Identities 41
 - 4. Cofunctions 44

 Calculating the Earth's Circumference 46

 EXERCISES 2 47

3. TRIGONOMETRIC FUNCTIONS OF REAL NUMBERS 49
 - A. TRIGONOMETRIC FUNCTIONS 49
 - 1. The Sine Function 49
 - 2. The Cosine Function 51
 - 3. The Tangent Function 53
 - 4. The Cotangent Function 55
 - 5. The Secant and Cosecant Functions 57
 - B. CALCULATING TRIGONOMETRIC VALUES ...60
 - 1. Trigonometric Values of Quadrantal Angles 60
 - 2. Using a Reference Angle 62
 - 3. Calculating Ratios from a Given Ratio 66
 - 4. Trigonometric Values of Other Angles 68
 - a. Using a trigonometric table 68
 - b. Using a calculator 71

 EXERCISES 3 73

4. TRIGONOMETRIC THEOREMS AND FORMULAS 76
 - A. TRIGONOMETRIC THEOREMS 76
 - 1. The Law of Cosines 76
 - 2. The Law of Sines 79
 - 3. Formulas for the Area of a Triangle 82
 - 4. Further Theorems (Optional) ...86
 - B. TRIGONOMETRIC FORMULAS 88
 - 1. Sum and Difference Formulas ...88
 - a. sin(x±y) 88
 - b. cos(x±y) 89
 - c. tan(x±y) 90
 - d. cot(x±y) 92
 - 2. Double-Angle and Half-Angle Formulas 93
 - a. sin 2x 93
 - b. cos 2x 95
 - c. tan 2x 96
 - d. cot 2x 96
 - e. Half-angle formulas 97
 - 3. Reduction Formulas 101
 - 4. Sum to Product Formulas 106
 - a. sin a ± sin b 106
 - b. cos a ± cos b 108
 - c. tan a ± tan b 110
 - d. cot a ± cot b 111
 - 5. Product to Sum Formulas 112
 - a. sin x · sin y 112
 - b. cos x · cos y 113
 - c. sin x · cos y 114
 - d. cos x · sin y 114

 Triangulation and Surveying 116

 EXERCISES 4 117

5. APPLIED TRIGONOMETRY 122

 EXERCISES 5 126

 CHAPTER SUMMARY 130

 CONCEPT CHECK 133

 CHAPTER REVIEW TEST 1 134

 CHAPTER REVIEW TEST 2 136

 CHAPTER REVIEW TEST 3 138

 ANSWERS 140

 GLOSSARY 143

INTRODUCTION

The branch of mathematics which we today call trigonometry developed from the study of the relationships between the sides and angles of a right triangle and also from the study of similar triangles. In the past there was no concept of trigonometric function, although people knew how to apply trigonometric properties to solve different problems.



Trigonometry was first used for cartography, astronomy, surveying and navigation. For example, the Egyptians used properties of similar triangles to help with land surveying and the construction of pyramids. The Babylonians also used trigonometry in their studies of astronomy, as an aid for the calculation of arcs of circles and of the lengths of the chords that subtended the arcs.

Late in the 8th century, Muslim astronomers inherited the knowledge mentioned above. By the end of the 10th century they had defined the sine function and the five other functions and had discovered and proved several basic theorems of trigonometry for both triangles in a flat plane and triangles on the surface of a sphere (these two types of triangle give us two types of trigonometry: plane trigonometry and spherical trigonometry). Several mathematicians suggested using $r = 1$ instead of $r = 60$; this exactly produces the modern values of the trigonometric functions. The Muslims also introduced the polar triangle in spherical trigonometry. All of these discoveries were used by astronomers as an aid in astronomical time-keeping and for finding the direction of Mecca for the five daily prayers required by Muslim law. Muslim scientists also produced very accurate trigonometric tables. For example, their tables of sine and tangent values, constructed for increments of 1/60 of a degree, were accurate for better than one part in 700 million.



Ebu'l Vefa Buzcani was the first scholar who put forward the basic rules of spherical and plane trigonometry. He analyzed the sine function scientifically and developed the following identities:

$$\sin(a+b) = \sin a \cdot \cos b + \cos a \cdot \sin b$$

$$\sin 2a = 2 \sin a \cdot \cos a, \text{ and}$$

$$\sin^2 a = 1 - \cos^2 a.$$

European scholars derived the same formulas using their own methods after many centuries and a lot of hard work. The great astronomer Nasir ad-Din at-Tusi wrote the *Book of the Transversal Figure*, which was the first treatment of plane and spherical trigonometry as an independent mathematical science.



The Western World became aware of Muslim trigonometry through the translations of Arabic astronomy handbooks, beginning in the 12th century. The first major Western work on trigonometry was written by the German astronomer and mathematician Johann Müller, who was known as Regiomontanus. In the next century the German astronomer Georges Joachim, known as Rheticus, introduced the modern concept of trigonometric functions as ratios instead of as the lengths of certain lines. It is believed that the term 'trigonometry' was first used by the German mathematician and astronomer Bartholomaeus Pitiscus who published a piece of work in 1595 that was later revised and called *Trigonometria sive de dimensione triangulae* in 1600. The word is derived from two Greek words, *trigonon* and *metria*, which mean 'triangle' and 'measurement' respectively. The French mathematician François Viète introduced the polar triangle into spherical trigonometry, and stated the multiple-angle formulas for sin ax and cos ax in terms of powers of sin x and cos x.



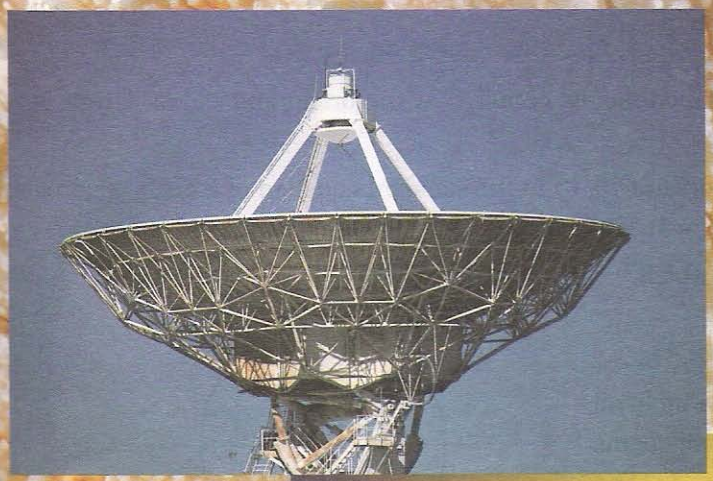
The Scottish mathematician John Napier also made important contributions to the development of trigonometry in the 17th century. Napier devised memory aids for the ten laws for solving spherical triangles, and developed some proportions called Napier's analogies for solving oblique spherical triangles. Almost exactly half a century after Napier's work was published, Isaac Newton invented differential and integral calculus. One of the foundations of this work was Newton's representation of different functions as infinite series of powers of x. Thus Newton developed a sries for sin x and similar series for cos x and tan x. With the invention of calculus, the trigonometric functions were taken over into analysis, where they still play important roles in both pure and applied mathematics. Finally, in the 18th century the Swiss mathematician Leonhard Euler defined the trigonometric functions in terms of complex numbers. This made the whole subject of trigonometry just one of the many applications of complex numbers, and showed that the basic laws of trigonometry were simply consequences of the arithmetic of these numbers. As a result, trigonometry has become an indispensable tool in many other areas of pure and applied mathematics.



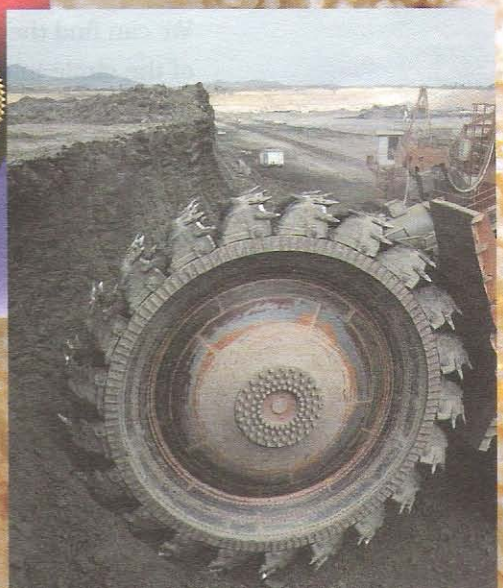
Today there are two main branches of trigonometry, namely plane trigonometry and spherical trigonometry. Plane trigonometry concerns triangles in a two dimensional plane only; it has many applications and is used in physics to calculate the properties of radiation, the propagation of light and other periodic phenomena. Spherical trigonometry deals with triangles on the surfaces of spheres. This form of trigonometry is mainly used in astronomy and for long-range navigation.

In this book we will begin our study of trigonometry by looking at plane trigonometry and its basic principles.

Introduction to TRIGONOMETRY







1 FUNDAMENTALS OF TRIGONOMETRY

A. THE UNIT CIRCLE



A circle is the set of points which are all the same distance from a fixed point in a plane. The fixed point is the center of the circle, and the distance of the points is called the radius of the circle.

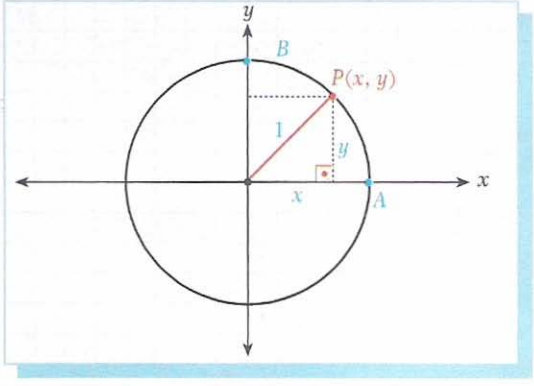
The **standard equation of a circle** with radius r and center $C(a, b)$ is $(x - a)^2 + (y - b)^2 = r^2$.

Definition unit circle

In the coordinate plane, a circle which has its center C at the origin and radius 1 unit is called a **unit circle**.

Substituting the coordinates of the origin $O(0, 0)$ in the standard equation of a circle gives us $(x - 0)^2 + (y - 0)^2 = 1^2$, i.e. $\mathbf{x^2 + y^2 = 1}$. This is the equation of a unit circle.

Thus we can write the set of points which form a unit circle as
$C = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } x^2 + y^2 = 1\}$.

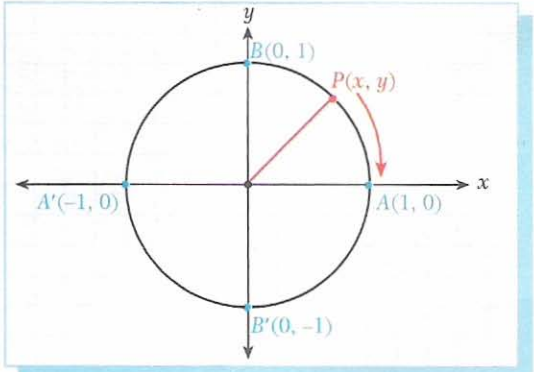


In other words, a unit circle is made up of all the points (x, y) whose x and y components satisfy the equation $x^2 + y^2 = 1$. The center of this circle is the origin and its radius is 1 unit.

We can find the x-intercepts and y-intercepts of the circle in the coordinate plane by using the unit circle equation $x^2 + y^2 = 1$.

For $y = 0$, $x^2 + 0^2 = 1$, so $x^2 = 1$ and $x = \pm 1$. So the x-intercepts of the unit circle are $A(1, 0)$ and $A'(-1, 0)$.

For $x = 0$, $0^2 + y^2 = 1$, so $y^2 = 1$ and $y = \pm 1$. So the y-intercepts of the unit circle are $B(0, 1)$ and $B'(0, -1)$.



EXAMPLE 1 $(a-3)x^2 + (b+1)y^2 = 1$ is the equation of a unit circle. Find the values of a and b.

Solution Since the equation of a unit circle is $x^2 + y^2 = 1$,

we have $(a-3)x^2 + (b+1)y^2 = x^2 + y^2$.

So $(a-3)x^2 = 1 \cdot x^2$; $(a-3) = 1$; $a = 4$.

Similarly, $(b+1)y^2 = 1 \cdot y^2$ and so

$(b+1) = 1$; $b = 0$.

EXAMPLE 2 Show that the point $P(\frac{1}{\sqrt{3}}, \frac{\sqrt{6}}{3})$ is on the unit circle.



Solution The equation of a unit circle is $x^2 + y^2 = 1$.

We have $x^2 + y^2 = \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{\sqrt{6}}{3}\right)^2$

$= \frac{1}{3} + \frac{6}{9}$

$= \frac{3}{9} + \frac{6}{9}$

$= \frac{9}{9}$

$= 1.$

So P is on the unit circle.

EXAMPLE 3 The point $P(x, \frac{\sqrt{3}}{2})$ is on the unit circle. Find the possible values of x.

Solution $x^2 + y^2 = 1$, so

$x^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$

$x^2 + \frac{3}{4} = 1$

$x^2 = 1 - \frac{3}{4}$

$x^2 = \frac{1}{4}$. So $x = \pm\frac{1}{2}$.

Try to imagine why we found two different values for x.

B. ANGLES AND DIRECTION

1. The Concept of Angle

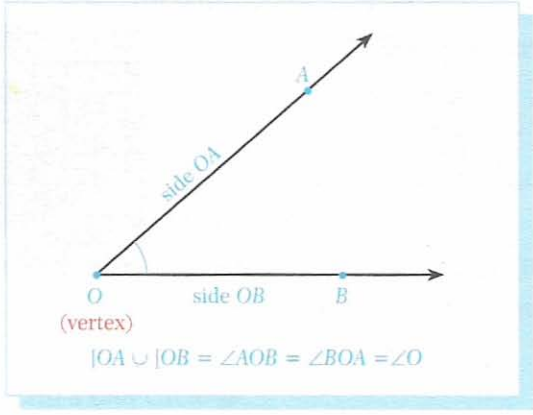
Definition

angle

An **angle** is the union of two rays which have a common endpoint.

The angle formed by the rays $[OA$ and $[OB$ is called **angle *AOB*** or **angle *BOA***. $[OA$ and $[OB$ are called the **sides** of the angle.

We can write the angle AOB as $[OA \cup [OB$, $\widehat{AOB}$ or $\angle AOB$. We can also write $[OA \cup [OB = \angle AOB$. The common endpoint O is called the **vertex** of the angle. If O is the vertex of one unique angle, we sometimes write $\angle O$ to or O the mean the angle with vertex O.

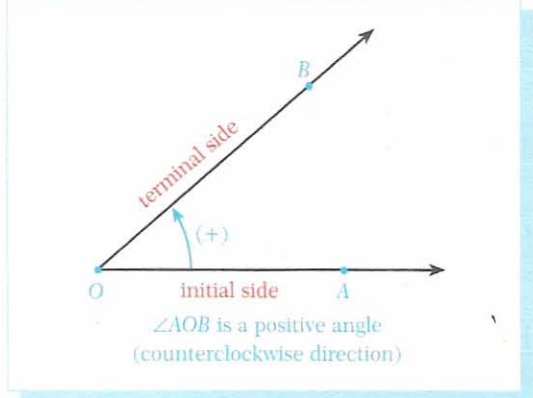


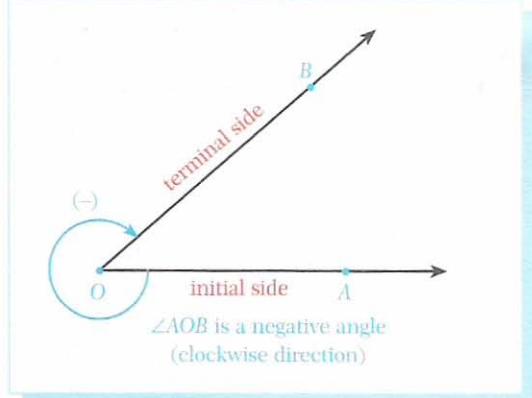


2. Directed Angles

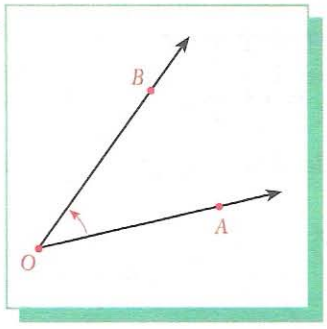
If we say that one of the two rays of an angle is the **initial side** of the angle and other side is the **terminal side** of the angle, then the corresponding angle is called a **directed angle**.

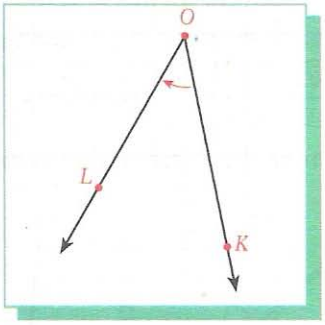
There are two different directions about the vertex of an angle from its initial side to its terminal side. If the angle is measured counterclockwise then the angle is called a **positive angle**. If it is measured clockwise then the angle is called a **negative angle**.

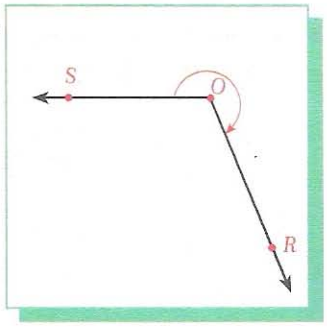




EXAMPLE 4 Determine the initial and terminal side of each angle and state its direction.

a. 

b. 

c. 

Solution

a. $[OA$ is the initial side and $[OB$ is the terminal side. $\angle AOB$ is a positive angle.

b. $[OK$ is the initial side and $[OL$ is the terminal side. $\angle KOL$ is a negative angle.

c. $[OS$ is the initial side and $[OR$ is the terminal side. $\angle SOR$ is a negative angle.

3. Directed Arcs

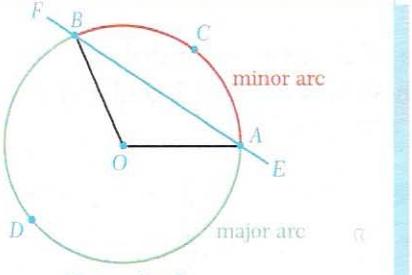
Definition

arc

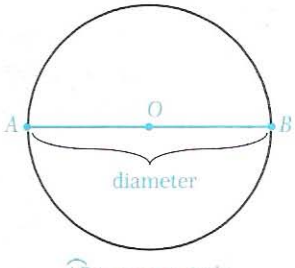
The segment of a circle between the two sides of an angle $\angle AOB$ is called the **arc** corresponding to $\angle AOB$. We write $\overset{\frown}{AB}$ to mean the arc corresponding to $\angle AOB$.

In order to distinguish the two arcs formed by the line AB on the circle, we can plot two points C and D as shown in the figure below. We denote the **positive arc** $\overset{\frown}{AB}$ by $\overset{\frown}{ACB}$ and the **negative arc** $\overset{\frown}{AB}$ by $\overset{\frown}{ADB}$.

The longer arc is called the **major arc** and the shorter arc is called the **minor arc**.



AB is a chord
EF is a secant line



$\overset{\frown}{AB}$ is a semicircle

A line which passes through a circle is called a **secant** line. A line segment AB which joins two different points on a **circle** is called a **chord**. Any secant or chord separates a circle into two arcs.

A chord which passes through the center of a circle is called a **diameter**. A diameter divides a circle into two equal arcs called **semicircles**.

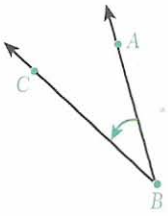
The length of every diameter in a given circle is the same. For this reason, when we talk about 'the diameter' of a circle, we mean the length of any diameter in the circle.

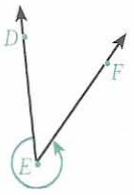
Check Yourself 1

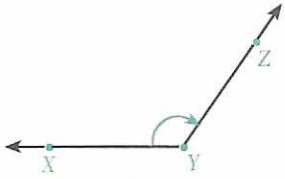
1. $6x^2 + (a + 2)y^2 = 2b + a$ is the equation of a unit circle. Find the values of a and b.
2. Determine whether or not each point lies on a unit circle.

 a. $P\left(\frac{\sqrt{2}}{2}, \frac{1}{\sqrt{2}}\right)$ b. $R(0, -1)$ c. $S(-\sqrt{3}, \sqrt{3})$ d. $T\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

3. Determine the initial and terminal side of each angle and indicate its direction.

a. 

b. 

c. 

Answers

1. $a = 4, b = 1$ 2. a. yes b. yes c. no d. yes
3. a. $[BA$ is the initial side and $[BC$ is the terminal side. Angle $\angle ABC$ is a positive angle.
 b. $[ED$ is the initial side and $[EF$ is the terminal side. Angle $\angle DEF$ is a positive angle.
 c. $[YX$ is the initial side and $[YZ$ is the terminal side. Angle $\angle XYZ$ is a negative angle.

C. UNITS OF ANGLE MEASURE

Recall that a **central angle** is an angle whose vertex is the center of a circle.

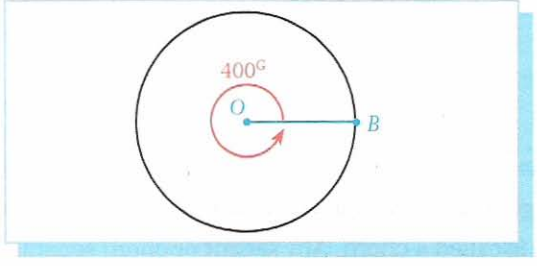
Definition complete angle

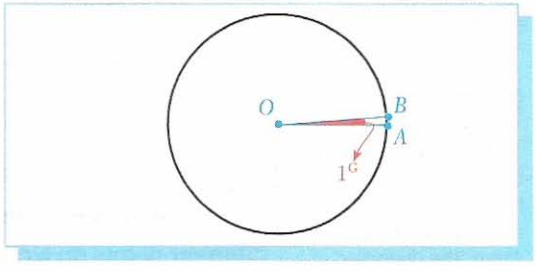
The central angle which corresponds to one complete revolution around a circle is called a **complete angle**.

1. Grad

Definition grad

When the circumference of a circle is divided into 400 equal parts, the central angle corresponding to one of these arcs is called **1 grad** and is denoted by 1^G. Thus the complete angle of a circle measures $400 \cdot 1^G = 400^G$.





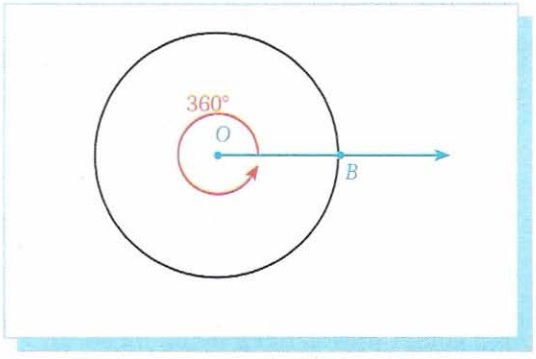
The grad unit was introduced in France, where it was called the *grade*, in the early years of the metric system.

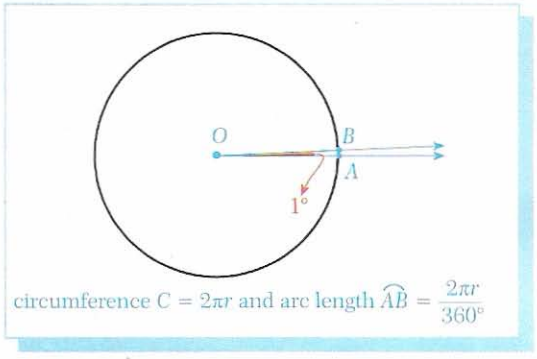
2. Degree

Definition

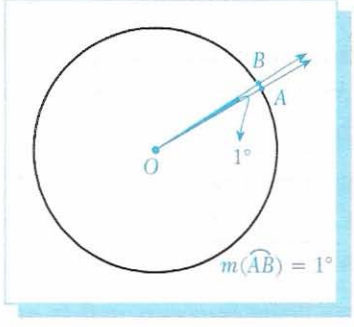
degree

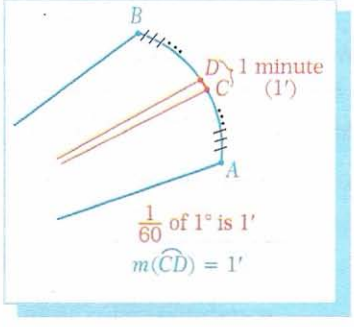
When the circumference of a circle is divided into 360 equal parts, the central angle corresponding to one of these arcs is called **1 degree** and is denoted by **1°**. Thus the measure of a complete angle is $360 \cdot 1° = 360°$.

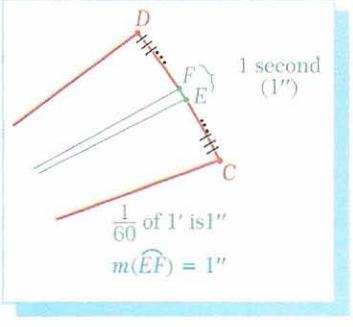




In order to measure smaller angles we use smaller angle units. Each degree can be divided into sixty equal parts called **minutes**, and each minute can be divided into sixty equal parts called **seconds**. 1 minute is denoted by $1'$ and 1 second is denoted by $1''$.







We can see that one degree is equal to $60 \cdot 60 = 3600$ seconds.

Now consider an angle which measures 37 degrees, 45 minutes and 30 seconds. We can write this angle in two ways:

in **degree-minute-second form**: $37° 45' 30''$.

in **decimal degree form**. We multiply the minutes by $\frac{1}{60}$ and the seconds by $\frac{1}{3600}$: $37° + (45 \cdot \frac{1}{60}) + (30 \cdot \frac{1}{3600}) = 37.758\overline{3}°$

EXAMPLE 5 a. Write 56° 20′ 15″ in decimal degree form.

b. Write 17.86° in degree-minute-second form.

Solution a. $56° 20' 15'' = \left(56 + 20 \cdot \frac{1}{60} + 15 \cdot \frac{1}{3600}\right)° = 56.3375°$

b.
$$\begin{aligned} 17.86° &= 17° + 0.86° && \text{(separate the decimal part)'} \\ &= 17° + 0.86 \cdot (60') && \text{(convert degrees to minutes)} \\ &= 17° + (51.60)' \\ &= 17° + 51' + 0.60' && \text{(separate the decimal part)} \\ &= 17° + 51' + 0.60 \cdot (60'') && \text{(convert minutes to seconds)} \\ &= 17° + 51' + 36'' \\ &= 17° 51' 36'' \end{aligned}$$

EXAMPLE 6 $x = 202° 15' 36''$ and $y = 114° 57' 58''$ are given. Perform the calculations.

a. $x + y$ b. $x - y$

Solution a. We begin by adding the degrees, minutes and seconds separately, starting with the seconds. We can see that the sum of the seconds is 36 + 58 = 94, which is greater than 60. Since 60″ = 1′ we can write 94″ = 60″ + 34″ = 1′ + 34″ = 1′ 34″. We add the extra 1 minute to the minutes part, so after adding the seconds the sum is 316° 73′ 34″.

$$\begin{array}{r} 202° 15' 36'' \\ +\ 114° 57' 58'' \\ \hline 316° 72' 94'' \end{array} \qquad \begin{array}{r} 202° 15' 36'' \\ +\ 114° 57' 58'' \\ \hline 316° 73' 34'' \end{array}$$

Now we have 73 minutes, which is also greater than 60. Since 60′ = 1°, 73′ = 60′ + 13′ = 1° + 13′ = 1° 13′. We add the extra 1 degree to the degrees part. The final result is 317° 13′ 34″, which is the sum of the angles.

$$\begin{array}{r} 202° 15' 36'' \\ +\ 114° 57' 58'' \\ \hline 316° 73' 34'' \end{array} \qquad \begin{array}{r} 202° 15' 36'' \\ +\ 114° 57' 58'' \\ \hline 317° 13' 34'' \end{array}$$

b. We subtract the degrees, minutes and seconds separately, starting with the seconds. However, we cannot subtract 58″ from 36″ directly. Instead, we carry 1 minute from the minutes part. 15′ becomes 14′ and the carried 1′ = 60″ is added to 36″: 60″ + 36″ = 96″. We can now subtract the seconds to get 96″ − 58″ = 38″.

$$\begin{array}{r} 202° 15' 36'' \\ -\ 114° 57' 58'' \\ \hline \end{array} \qquad \begin{array}{r} 202° 14' 96'' \\ -\ 114° 57' 58'' \\ \hline 38'' \end{array}$$

Similarly, we cannot subtract 57′ from 14′ directly. We carry 1 degree from the degrees part so 202° becomes 201° and the carried 1° = 60′ is added to 14′:

60′ + 14′ = 74′. We can now subtract the minutes to get 74′ – 57′ = 17′.

$$\begin{array}{r} 202^\circ\ 14'\ 96'' \\ -\ 114^\circ\ 57'\ 58'' \\ \hline 38'' \end{array} \qquad \begin{array}{r} 201^\circ\ 74'\ 96'' \\ -\ 114^\circ\ 57'\ 58'' \\ \hline 17'\ 38'' \end{array}$$

Finally we subtract the degrees: 201° – 114° = 87°.

The final result is 87° 17′ 38″, which is the difference of the angles.

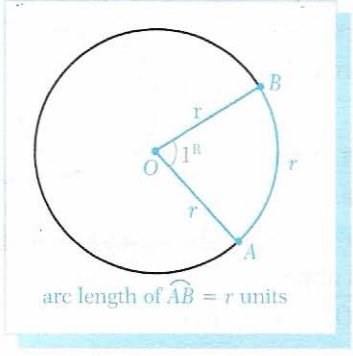
$$\begin{array}{r} 201^\circ\ 74'\ 96'' \\ -\ 114^\circ\ 57'\ 58'' \\ \hline 87^\circ\ 17'\ 38'' \end{array}$$

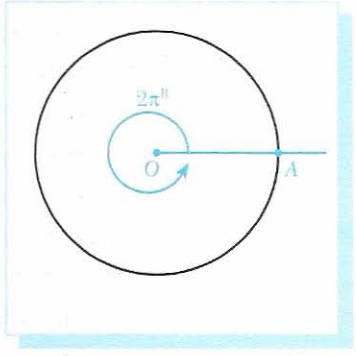
3. Radian

Definition

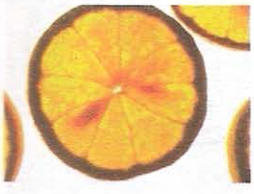
radian

Let $\overset{\frown}{AB}$ be an arc of a circle with radius r such that the arc length of $\overset{\frown}{AB}$ is also r. Then the measure of the central angle corresponding to $\overset{\frown}{AB}$ is called **1 radian**. It is denoted by **1 rad** or 1^R.

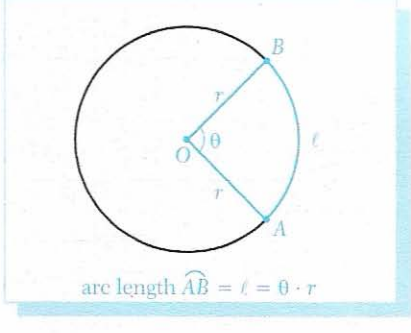




We know that the circumference C of a circle can be calculated using the formula $C = 2\pi r$. So C consists of 2π times an arc with length r. Therefore the measure of the central angle corresponding to the complete arc of a circle is **2π radians**.

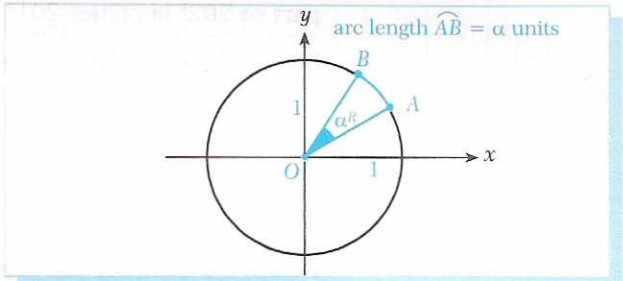


The radian measure gives us an easy correspondence between the length of an arc, the radius r of its circle and the measure of the central angle corresponding to the arc. If the arc length is r, the angle measure is one radian. If the arc length is ℓ, the angle measure is $\frac{\ell}{r}$ radians. In the figure opposite, $m(\angle AOB) = \theta^R$ and arc length $\overset{\frown}{AB} = \ell$, so $\theta = \frac{\ell}{r}$ and $\ell = r \cdot \theta$.

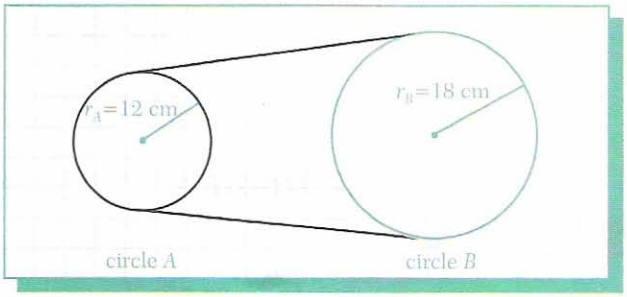


Remark 1

On a unit circle, the measure of an angle in radians is equal to the length of the corresponding arc.

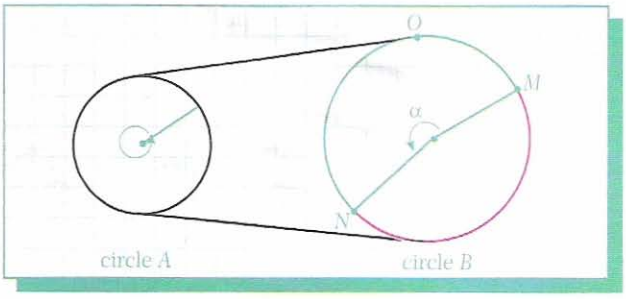


EXAMPLE 7 Two circles are connected with a string as shown in the figure. When the small circle makes one full rotation, through how many radians will the big circle rotate?



Solution In the figure, $\overset{\frown}{MON}$ shows the distance that the big circle rotates. The circumference of the small circle A is $C_A = 2\pi \cdot 12 = 24\pi$ cm.

After one complete rotation of circle A, circle B rotates through the same arc length as C_A, so arc length $\overset{\frown}{MON} = 24\pi$ cm. The central angle α corresponding to the arc $\overset{\frown}{MON}$ is the angle of rotation of circle B.



Since $\overset{\frown}{MON} = 24\pi$ and $r_B = 15$ cm, we have $\alpha = \dfrac{24\pi}{18}$, i.e. $\alpha = \dfrac{4\pi^R}{3}$. So the big circle rotates through $\dfrac{4\pi^R}{3}$.

Remember!
Complementary angles add up to 90°.
Supplementary angles add up to 180°.

Check Yourself 2

1. Find the complementary and supplementary angles of 66° 38′ 11″.
2. x = 47° 23′ 05″ and y = 32° 04′ 27″ are given. Find $2x + 3y$.

Answers

1. complementary: 23° 21′ 49″
 supplementary: 113° 21′ 49″
2. 190° 59′ 31″

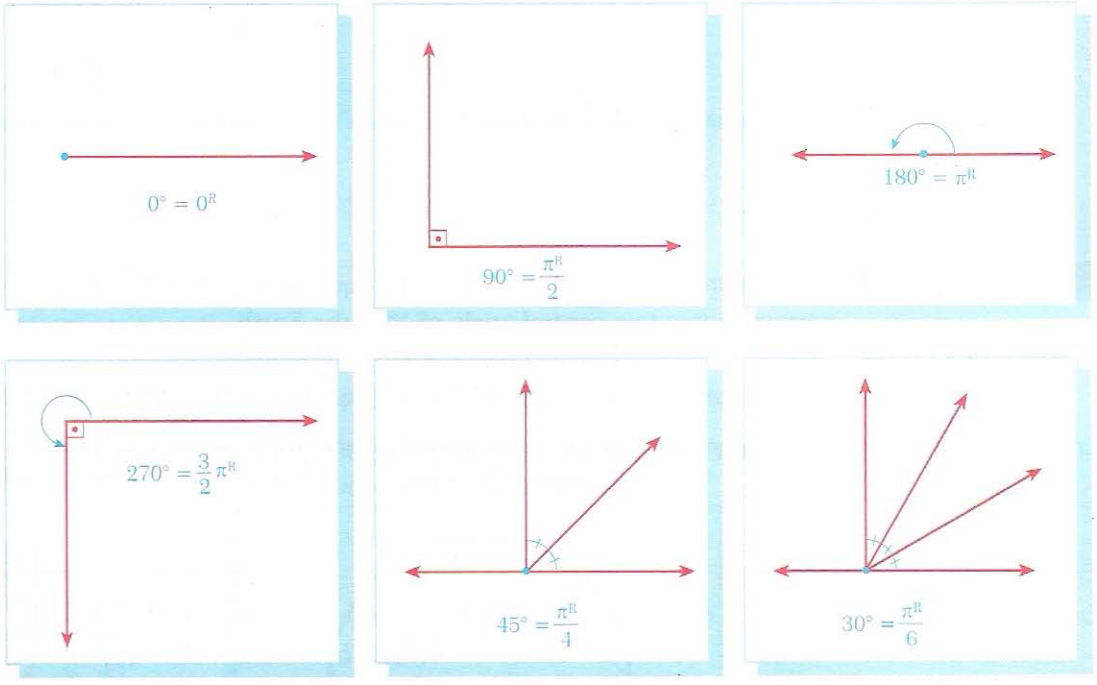


4. Converting Units of Angle Measure

We have seen that the complete angle of a circle measures $360° = 2\pi^R$.

We can use the formula $\frac{D}{180°} = \frac{R}{\pi^R}$ to relate degree and radian measurements, where D and R represent the degree and radian measurements respectively.

This formula gives us the following relations between common angle measures:



EXAMPLE 8 Convert the angle measures.

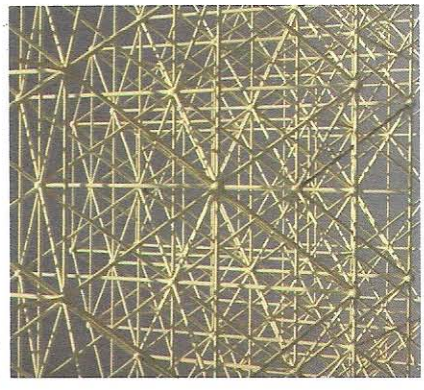
a. $100°$ to radians b. $\frac{5\pi^R}{12}$ to degrees

Solution

a. For $D = 100°$ the formula gives us

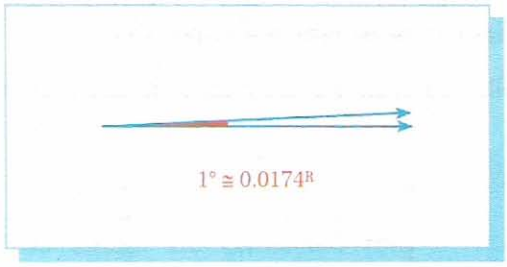
$$\frac{100°}{180°} = \frac{R}{\pi^R}, \text{ so } R = \frac{5\pi^R}{9}.$$

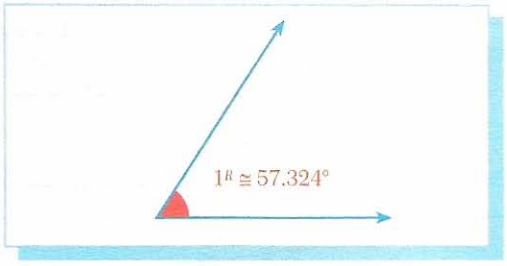
b. For $R = \frac{5\pi^R}{12}$ we have $\frac{D}{180°} = \frac{\frac{5\pi^R}{12}}{\pi^R}$, so $D = 75°$.



The formula also gives us the following results ($\pi \cong 3.14$):

$$1^\circ = \frac{\pi}{180} \cong 0.0174^R \text{ and } 1^R = \frac{180^\circ}{\pi} \cong 57.325^\circ$$

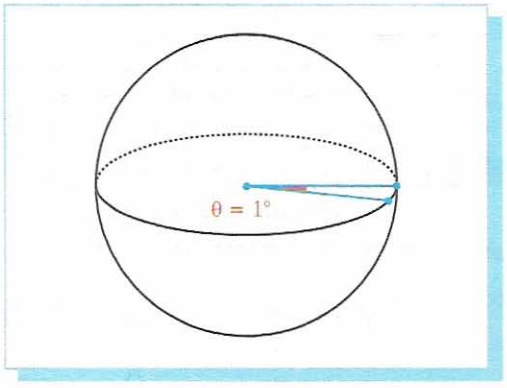


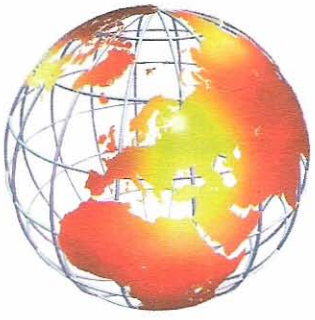


EXAMPLE 9 The circumference of the Earth at the equator is 40 000 km. Find the length of the equator corresponding to a central angle of 1° of the Earth.

Solution The complete central angle which corresponds to the total length of the equator is 360°.

Since 360° corresponds to the length 40 000 km, 1° corresponds to $\frac{40\,000}{360} \cong 111$ km.





Note

The **rad** unit or exponential ^{R} notation is often omitted when we write an angle measure in radians. If no unit is specified for an angle measure, this means that the measure is in radians. For example, the statement $\alpha = \frac{\pi}{2}$ means that the angle α measures $\frac{\pi}{2}$ radians, not $\frac{\pi}{2}$ degrees. From now on in this module, if we do not give a unit for an angle measure then the measure is in radians.

Check Yourself 3

1. Convert the measures to radians.

 a. 30° b. 135° c. 210° d. 900°

2. Convert the measures to degrees.

 a. $\frac{\pi}{3}$ b. $\frac{5\pi}{6}$ c. $\frac{5\pi}{4}$ d. 10π

Answers

1. a. $\frac{\pi}{6}$ b. $\frac{3\pi}{4}$ c. $\frac{7\pi}{6}$ d. 5π
2. a. 60° b. 150° c. 225° d. 1800°

D. PRIMARY DIRECTED ANGLES

1. Coterminal Angles

Definition

standard position of an angle

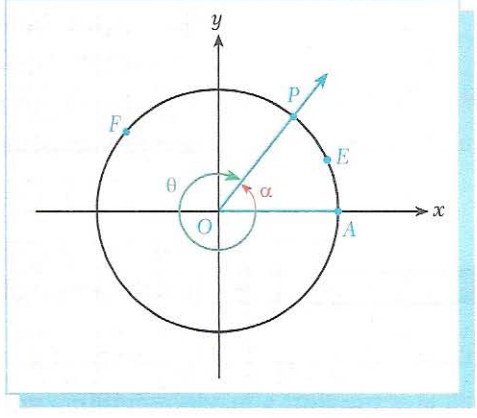
An angle in the coordinate plane whose vertex is at the origin and whose initial side lies along the positive x-axis is said to be in **standard position.**

Definition

coterminal angles

Two or more angles whose terminal sides coincide with each other when they are in standard position are called **coterminal** angles.

Let us look at an example of coterminal angles. The figure shows a unit circle.



The positive angle $\angle AOP$ corresponds to the arc $\overset{\frown}{AEP}$ and the negative angle $\angle AOP$ corresponds to the arc $\overset{\frown}{AFP}$. These angles are coterminal. The measure of the positive angle $\angle AOP$ is $m(\angle AOP) = \alpha°$ and the measure of the negative angle $\angle AOP$ is $m(\angle AOP) = -(360 - \alpha)°$.

We can also express the measure of each angle in radians. Since this is a unit circle, if the length of the arc $\overset{\frown}{AEP}$ is θ then the measure of the positive angle $\angle AOP$ is $m(\angle AOP) = \theta$ and the measure of the negative angle $\angle AOP$ is $m(\angle AOP) = -(2\pi - \theta)$.

Now assume that point P in the figure is moving around the circumference of the unit circle from point A in the counterclockwise direction. Study the following table.

Position of point P (moving counterclockwise)	Measure of the central angle for $\overset{\frown}{AP}$	
	Degrees	Radians
P lies on the positive x-axis	$0°$	0
P lies on the positive y-axis	$90°$	$\frac{\pi}{0}$
P lies on the negative x-axis	$180°$	π
P lies on the negative y-axis	$270°$	$\frac{3\pi}{2}$
P lies on the positive x-axis after one complete revolution	$360°$	2π
P lies on the ray $[OP$ after one complete revolution	$360° + \alpha$	$2\pi + \theta$
P lies on the ray $[OP$ after a second revolution	$720° + \alpha$	$4\pi + \theta$
P lies on the ray $[OP$ after its k^{th} revolution	$k \cdot 360° + \alpha$	$2k\pi + \theta$

EXAMPLE 10 For each angle, write the set of coterminal angles with the same unit of measurement.

a. $175°$ b. $\frac{5\pi}{4}$

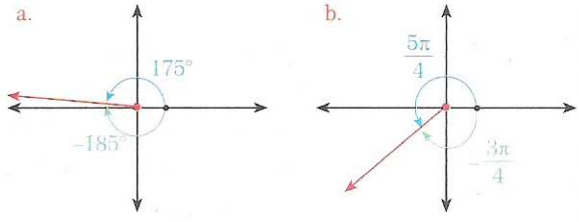
Solution Coterminal angles differ by an integral multiple of complete angles.

a. $\{175° + k \cdot 360°, k \in \mathbb{Z}\} = \{\ldots,-545°, -185°, 175°, 535°, 895°,\ldots\}$

b. $\{\frac{5\pi}{4} + k \cdot 2\pi, k \in \mathbb{Z}\} = \{\ldots, -\frac{11\pi}{4}, -\frac{3\pi}{4}, \frac{5\pi}{4}, \frac{13\pi}{4}, \frac{21\pi}{4}, \ldots\}$

Note

The angles in part a and part b are coterminal. Therefore, if we graph them in standard position, these angles will have the same terminal side.



EXAMPLE 11 Find the arc length which corresponds to the central angle $40°$ on the unit circle ($\pi \cong 3$).

Solution Since $\frac{D}{180} = \frac{R}{\pi}$ we have $\frac{40°}{180} = \frac{R}{\pi}$, so $R = \frac{2\pi}{9}$. We know that on the unit circle, the radian measure of a directed angle is equal to the length of the directed arc corresponding to the angle. So the arc length is $\frac{2\pi}{9}$, and using $\pi \cong 3$ gives us $\frac{2\pi}{9} \cong \frac{2 \cdot 3}{9} = \frac{2}{3} \cong 0.\overline{6}$.

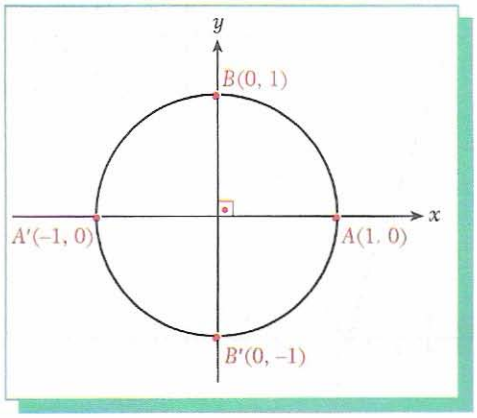
EXAMPLE 12 Find the coordinates of the terminal point of the arc with length $\frac{\pi}{2}$ which is in standard position on the unit circle.

Solution The circumference of a unit circle measures 2π.

So $\frac{\pi}{2}$ represents a quarter of the circle.

Therefore the arc length $\frac{\pi}{2}$ corresponds to the point $B(0, 1)$ on the unit circle.

Furthermore, the arc length π corresponds to the point $A'(-1, 0)$ on the unit circle and $\frac{3\pi}{2}$ corresponds to $B'(0, -1)$.



2. Primary Directed Angles and Arcs

Definition

primary directed angle

Let β be an angle which is greater than 360°. Then the positive angle $\alpha \in [0, 360°)$ which is coterminal with β is called the **primary directed angle** of β.

In order to find a primary directed angle α we must divide the initial angle by 360. We must not simplify before the division, because 360 represents a complete rotation. For example, the remainder in the operation 5000÷360 gives us the required primary directed angle whereas the simplified version 500÷36 does not.

In other words, the primary directed angle of β is the smallest positive angle that is coterminal with β. If we divide β by 360°, the remainder will be the primary directed angle.

$$m(\beta) = k \cdot (360°) + m(\alpha), \; k \in \mathbb{Z}.$$

For example, 30° is the primary directed angle of 390° because $390° = 1 \cdot 360° + 30°$.

We know that the radian measure of any angle is equal to the length of the arc which corresponds to its central angle in the unit circle. The circumference of a unit circle is 2π. Therefore any two real numbers that differ by integral multiples of 2π will coincide at the same point on the circle.

Definition primary directed arc

The positive real number $t \in [0, 2\pi)$ which differs from a real number by integral multiples of 2π is called a **primary directed arc.**

Since t is the smallest positive real number that is coterminal with a given angle θ, we can find t by subtracting integral multiples of complete rotations from θ, or alternatively by dividing θ by 2π and considering the remainder:

$$\theta = k \cdot (2\pi) + t,\ k \in \mathbb{Z}.$$

EXAMPLE 13 Find the primary directed angle of each angle, using the same unit.

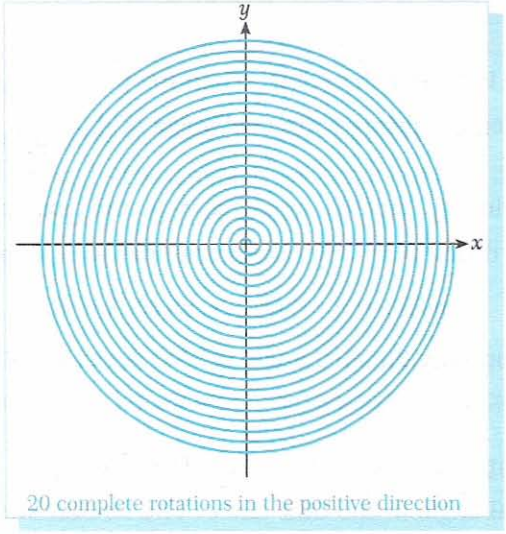
a. $7320°$ b. $-7320°$ c. $\frac{75\pi}{8}$ d. $-\frac{75\pi}{8}$

Solution a.

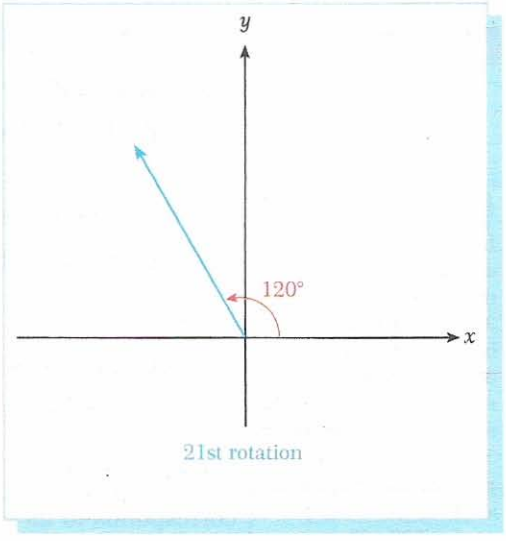
$$\begin{array}{r|l} 7320 & 360 \\ \hline -\ 7200 & 20 \\ \hline 120 & \end{array}$$

number of rotations





20 complete rotations in the positive direction



21st rotation

$7320° = (20 \cdot 360°) + 120°$, so $120°$ is coterminal with $7320°$.

So the primary directed angle of $7320°$ is $120°$.

b. Solution 1

$7320° = (20 \cdot 360°) + 120°$

$-7320° = -(20 \cdot 360° + 120°)$

$= (-20 \cdot 360°) - 120°$

$= -120°$

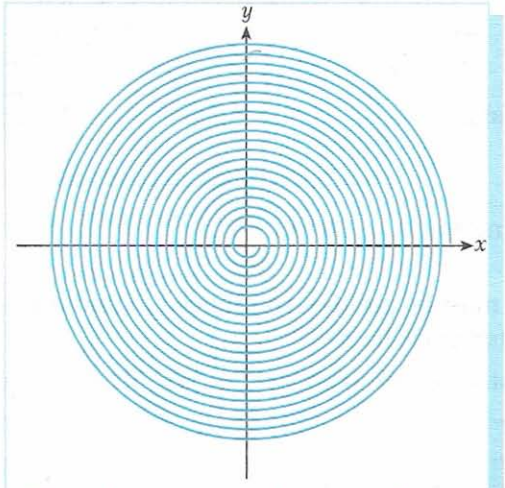
$-120° \equiv 240°$ (coterminal angles)

$-7320° \equiv 240°$ (coterminal angles)

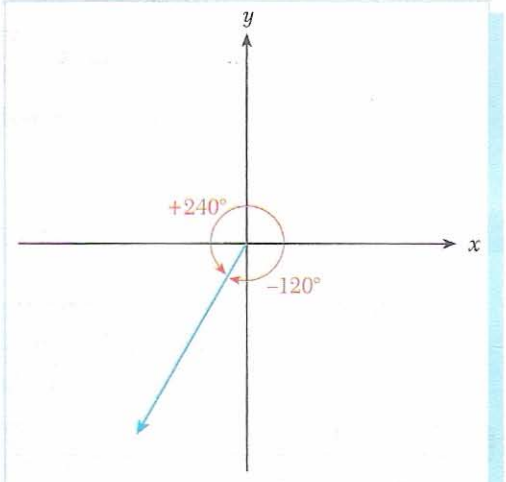
Solution 2

$-7320° = (-21) \cdot 360° + 240°$

Therefore the primary directed angle of $-7320°$ is $240°$.



20 complete rotations in the negative direction



21st rotation

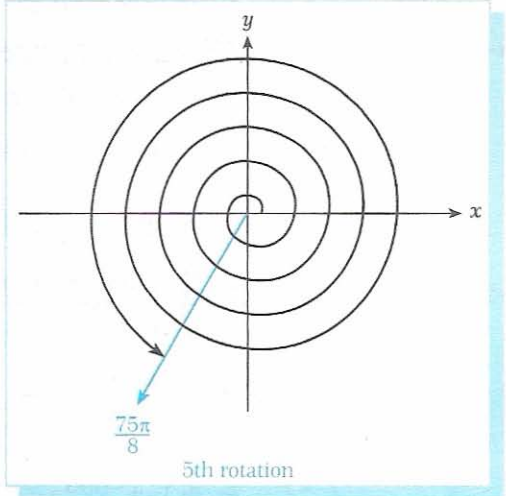
c. Solution 1

$$\frac{75\pi}{8} = \frac{64\pi}{8} + \frac{11\pi}{8}$$

$$\frac{75\pi}{8} = 4 \cdot (2\pi) + \frac{11\pi}{8}$$

number of rotations

$$\alpha = \frac{11\pi}{8}$$



5th rotation

Solution 2

1. Divide the numerator by twice the denominator:

 $2 \cdot 8 = 16$ and $75 \div 16 = (4 \cdot 16) + 11$.

2. Multiply the remainder by

 $$\frac{\pi}{\text{denominator}}: \quad 11 \cdot \frac{\pi}{8} = \frac{11\pi}{8}.$$

So the primary directed angle of $\frac{75\pi}{8}$ is $\frac{11\pi}{8}$.

$$\begin{array}{r|l} 75 & 16 \\ \hline -\ 64 & 4 \\ \hline 11 & \end{array}$$

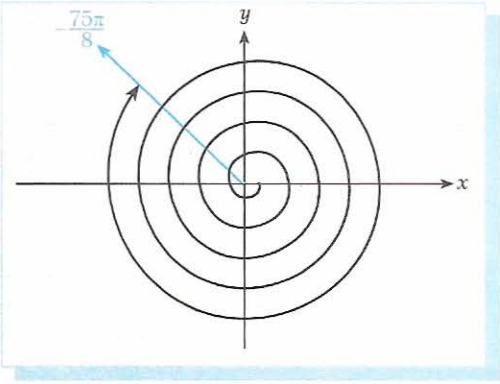
4 → number of rotations

11 → remainder

d. If the angle was positive, the remainder would be $\frac{11\pi}{8}$ as we found in part c. But the angle is negative, so the remainder is $-\frac{11\pi}{8}$.

Because a coterminal angle must be positive, we calculate $-\frac{11\pi}{8} \equiv 2\pi - \frac{11\pi}{8} = \frac{5\pi}{8}$.

So the primary directed angle is $\frac{5\pi}{8}$.



EXAMPLE 14 Find the primary directed angle of $\theta = -30°\ 42'\ 15''$.

Solution The primary directed angle must be positive, so we need to find the positive difference from 360°. Let the primary directed angle be θ'.

To make the calculation easier we can write 360° as 359° 59′ 60″. Then

$1° = 60'$ and $1' = 60''$ so $360° = 359°\ 59'\ 60''$.

$$\begin{array}{r} 359°\ 59'\ 60'' \\ -\ \ 30°\ 42'\ 15'' \\ \hline 329°\ 17'\ 45''. \end{array}$$

So $\theta' = 329°\ 17'\ 45''$.

Check Yourself 4

1. Find the primary directed angle of each angle, using the same unit of measurement.

 a. 100° b. 7200° c. $\frac{33\pi}{5}$ d. $\frac{3\pi}{2}$

 e. −400° f. −50° g. $-\frac{11\pi}{3}$ h. $-\frac{5\pi}{4}$

2. For each angle, write the set of coterminal angles with the same unit of measurement.

 a. 30° b. 120° c. $\frac{\pi}{3}$ d. $\frac{3\pi}{2}$

Answers

1. a. 100° b. 0° c. $\frac{3\pi}{5}$ d. $\frac{3\pi}{2}$ e. 320° f. 310° g. $\frac{\pi}{3}$ h. $\frac{3\pi}{4}$

2. a. {..., −690°, −330°, 30°, 390°, ...} b. {..., −600°, −240°, 120°, 480°, 840, ...}

 c. $\{..., -\frac{11\pi}{3}, -\frac{5\pi}{3}, \frac{\pi}{3}, \frac{7\pi}{3}, \frac{13\pi}{3}, ...\}$ d. $\{..., -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, ...\}$

EXERCISES 1

A. The Unit Circle

1. Find the ordered pair (a, b) that makes each equation a unit circle equation.

 a. $(a + 1)x^2 + (b - 2)y^2 = 1$

 b. $(2a + 5)x^2 + (1 - 4b)y^2 = 9$

 c. $x^2 + y^2 + ax + (b - 1)y - 1 = 0$

 d. $2x^2 + 2y^2 + ax + by = 2$

2. The points below are on the unit circle. Find the unknown coordinate in each point.

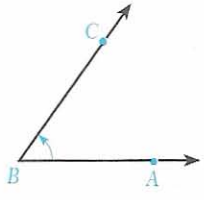
 a. $A\left(\frac{1}{2}, y\right)$ b. $B\left(\frac{-\sqrt{3}}{2}, y\right)$

 c. $C\left(x, \frac{1}{3}\right)$ d. $D\left(x, \frac{\sqrt{2}}{2}\right)$

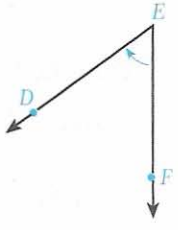
B. Angles and Direction

3. Determine the initial and the terminal sides of each angle.

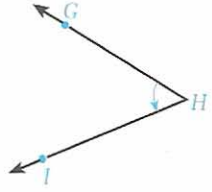
 a.



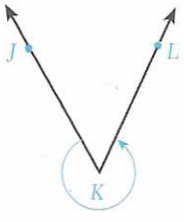
 b.



 c.

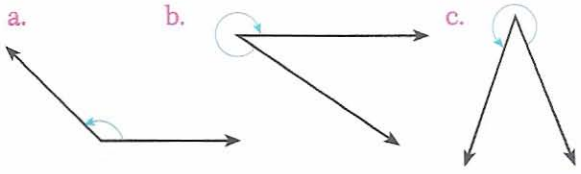


 d.



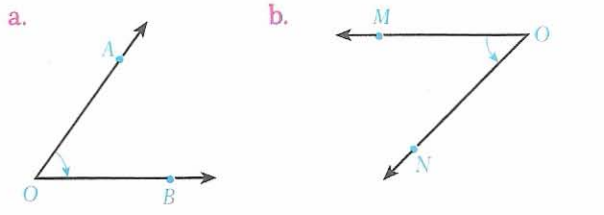
4. Determine the direction (positive or negative) of each angle.

 a. b. c.

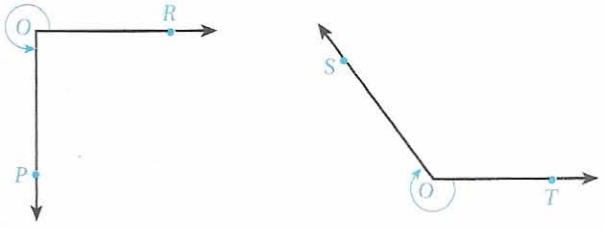


5. Name each angle and state its direction.

 a. b.

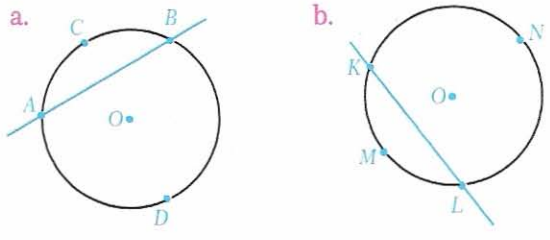


 c. d.



6. Name the major arc and the minor arc on each circle.

 a. b.



C. Units of Angle Measure

7. Find the degree measure of the angle rotation through the given number of revolutions.

 a. $\frac{3}{4}$ revolution clockwise

 b. $\frac{2}{3}$ revolution counterclockwise

 c. $\frac{10}{3}$ revolutions clockwise

 d. $\frac{9}{4}$ revolutions counterclockwise

8. Find the radian measure of the angle rotation through the given number of revolutions.

 a. $\frac{1}{4}$ revolution clockwise

 b. $\frac{3}{5}$ revolution counterclockwise

 c. $\frac{9}{4}$ revolutions clockwise

 d. $\frac{10}{3}$ revolutions counterclockwise

9. Write each angle in decimal degree form.

 a. 10° 45′ b. 80° 15′

 c. 37° 21′ 30″ d. 89° 59′ 60″

10. Write each angle in degree-minute-second form.

 a. 10.10° b. 82.15°

 c. 54.30° d. 23.73°

11. Perform the operations.

 a. 72° 10′ 20″ + 30° 40′ 25″

 b. 42° 10′ 23″ – 18° 20′ 35″

 c. –55° 07′ 53″ – 50° 15′ 03″

 d. 2 · (14° 15′ 17″) + 3 · (73° 07′ 10″)

12. Complete the table with the angle measures.

Degrees	60°	b	210°	d
Radians	a	$\frac{3\pi}{4}$	c	$\frac{11\pi}{6}$

13. Convert the angle measures (degrees to radians and radians to degrees).

 a. 100° b. 250°

 c. 1200° d. –3000°

 e. $\frac{\pi}{12}$ f. $\frac{17\pi}{18}$

 g. $\frac{201\pi}{4}$ h. $-\frac{17\pi}{6}$

D. Primary Directed Angles

14. Find the primary directed angle of each angle, using the same unit.

 a. 1234° b. –4321°

 c. $\frac{190\pi}{9}$ d. $-\frac{90\pi}{19}$

15. An arc lies in standard position on the unit circle. Find the coordinates of the terminal point of the arc if the arc has length

 a. π. b. $\frac{5\pi}{4}$.

16. Find the primary directed angle of 1720 grads in degrees.

17. Solve for x if $x \in \mathbb{R}$, $k \in \mathbb{Z}$.

 a. $2x - 120° = 90° + (k \cdot 360°)$

 b. $\frac{x}{3} + \frac{\pi}{4} = \frac{\pi}{3} + \frac{x}{4} + (k \cdot 2\pi)$

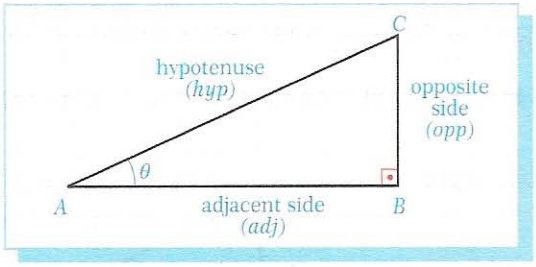
 c. $4x = k \cdot 360°$

 d. $3x - 150° = \frac{\pi}{6} + (k \cdot 2\pi)$

2 RIGHT TRIANGLE TRIGONOMETRY

A. TRIGONOMETRIC RATIOS

1. Definition

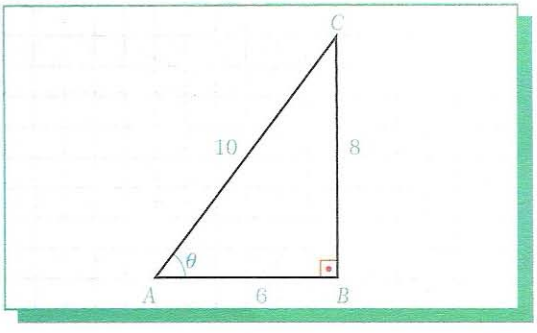




Consider the right triangle in the figure. The table shows the trigonometric ratios for the acute angle θ.

Ratio name	Ratio abbreviation	Ratio definition	Abbreviated definition
sine	$\sin\theta$	$\frac{\text{length of side opposite } \theta}{\text{length of hypotenuse}}$	$\frac{opp}{hyp}$
cosine	$\cos\theta$	$\frac{\text{length of side adjacent to } \theta}{\text{length of hypotenuse}}$	$\frac{adj}{hyp}$
tangent	$\tan\theta$	$\frac{\text{length of side opposite } \theta}{\text{length of side adjacent to } \theta}$	$\frac{opp}{adj}$
cotangent	$\cot\theta$	$\frac{\text{length of side adjacent to } \theta}{\text{length of side opposite } \theta}$	$\frac{adj}{opp}$
secant	$\sec\theta$	$\frac{\text{length of hypotenuse}}{\text{length of side adjacent to } \theta}$	$\frac{hyp}{adj}$
cosecant	$\csc\theta$	$\frac{\text{length of hypotenuse}}{\text{length of side opposite } \theta}$	$\frac{hyp}{opp}$

EXAMPLE 15 The figure shows a right triangle. Write the six trigonometric ratios for the angle θ.



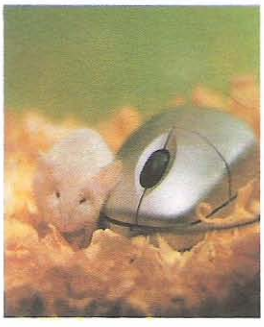
Solution $\sin\theta = \frac{opp}{hyp} = \frac{4}{5}$ $\cos\theta = \frac{adj}{hyp} = \frac{3}{5}$ $\tan\theta = \frac{opp}{adj} = \frac{4}{3}$

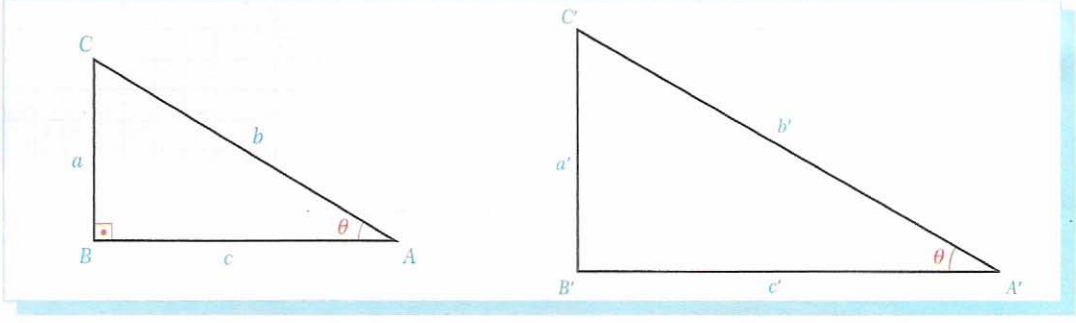
$\cot\theta = \frac{adj}{opp} = \frac{3}{4}$ $\sec\theta = \frac{hyp}{adj} = \frac{5}{3}$ $\csc\theta = \frac{hyp}{opp} = \frac{5}{4}$

We know that any ratio can be expanded or simplified by multiplying its numerator and denominator by the same non-zero number. For example:

$\frac{2}{3} = \frac{2k}{3k} = \frac{4}{6} = \frac{40}{60} = \frac{200}{300} = \ldots$ etc. where k is any non-zero number.

This property is also used in trigonometry. Look at the two right triangles below.

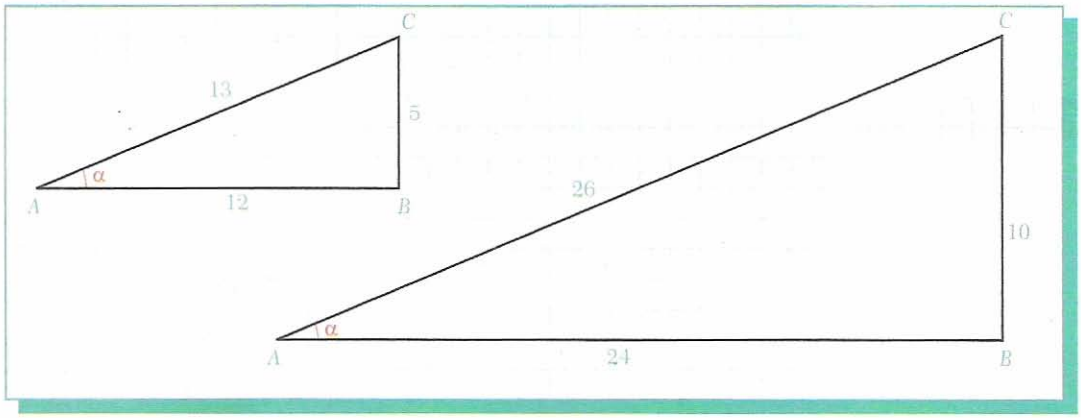




$\sin\theta = \frac{a}{b}$ $\sin\theta = \frac{a'}{b'}$

Although the lengths of the sides of the triangles are different, the two trigonometric ratios for the common angle θ are the same: $\frac{a}{b} = \frac{a'}{b'}$. In other words, the sides are in proportion. We say that these triangles are **similar**.

EXAMPLE 16 Two right triangles are shown below. Find the trigonometric ratios for the angle α in each triangle and show that they are equal.



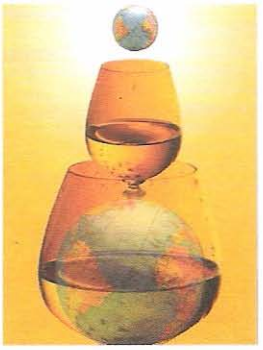
Solution

$$\sin\alpha = \frac{opp}{hyp} = \frac{5}{13} \qquad \sin\alpha = \frac{opp}{hyp} = \frac{10}{26} = \frac{5}{13}$$

$$\cos\alpha = \frac{adj}{hyp} = \frac{12}{13} \qquad \cos\alpha = \frac{adj}{hyp} = \frac{24}{26} = \frac{12}{13}$$

$$\tan\alpha = \frac{opp}{adj} = \frac{5}{12} \qquad \tan\alpha = \frac{opp}{adj} = \frac{10}{24} = \frac{5}{12}$$

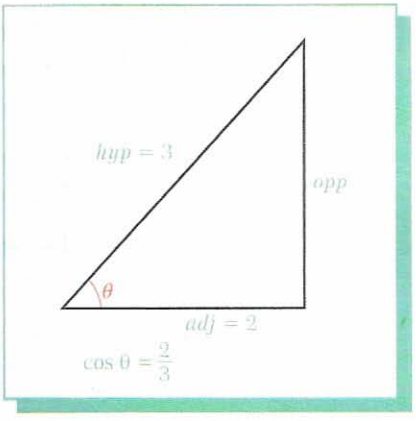
$$\cot\alpha = \frac{adj}{opp} = \frac{12}{5} \qquad \cot\alpha = \frac{adj}{opp} = \frac{24}{10} = \frac{12}{5}$$



The ratios are the same because the sides are in proportion: these are similar triangles.

EXAMPLE 17 In a right triangle, θ is an acute angle such that $\cos\theta = \frac{2}{3}$. Find the sine, tangent and cotangent ratios of the same angle.

Solution We do not know the lengths of the sides of the triangle. However, we know that any right triangle with angle θ will be similar to this triangle. So we can use the numerator and denominator of the given ratio $(\frac{2}{3})$ as two sides of the triangle, as shown in the figure.



Now we can use the Pythagorean Theorem to find the length of the opposite side:

$$opp^2 + 2^2 = 3^2$$
$$opp^2 + 4 = 9$$
$$opp^2 = 5 \text{ so } opp = \sqrt{5}.$$

The resulting right triangle gives us the following results:

$$\sin\theta = \frac{opp}{hyp} = \frac{\sqrt{5}}{3};\quad \tan\theta = \frac{opp}{adj} = \frac{\sqrt{5}}{2};\quad \cot\theta = \frac{adj}{opp} = \frac{2}{\sqrt{5}}.$$

Check Yourself 5

1. In a right triangle, θ is an acute angle such that $\tan\theta = 4$. Find the sine, cosine and cotangent ratios of the same angle.
2. One leg of an isosceles right triangle is 1 unit long. Find all the trigonometric ratios of one of the two equal acute angles in the triangle.

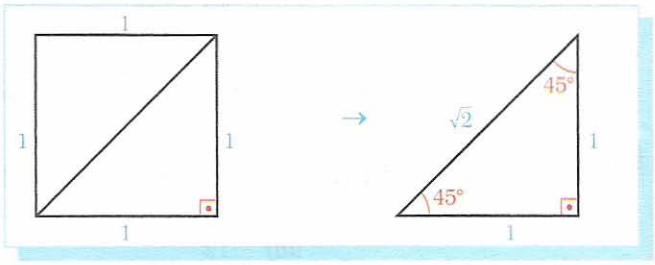
Answers

1. $\sin\theta = \frac{4}{\sqrt{17}}$, $\cos\theta = \frac{1}{\sqrt{17}}$, $\cot\theta = \frac{1}{4}$

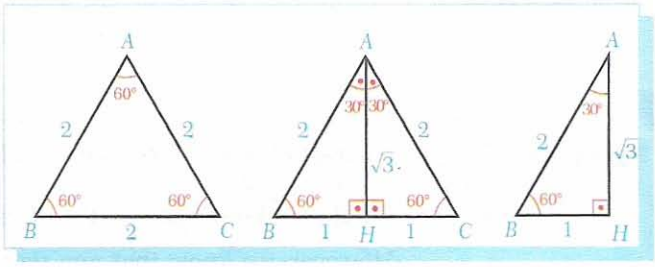
2. $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$, $\tan 45° = \cot 45° = 1$, $\csc 45° = \sec 45° = \sqrt{2}$

2. Special Triangles and Ratios

Certain right triangles have ratios which we can calculate easily using the Pythagorean Theorem. One example is the isosceles right triangle which we obtain when we bisect a square diagonally. If the square has side length 1 unit we obtain the isosceles right triangle shown in the figure.



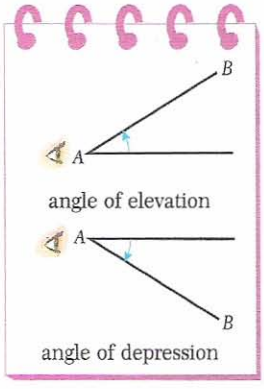
Another example is the right triangle which we obtain when we bisect of an equilateral triangle from an altitude to a base. If the triangle has side length 2 units we obtain the right triangle shown in the figure.

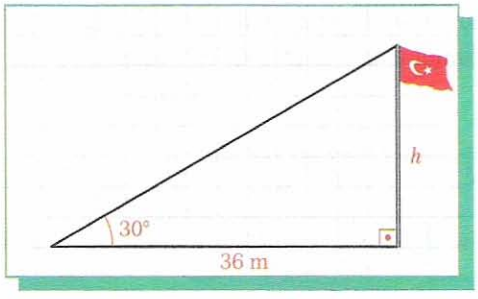


In each example we can find the unknown length using the Pythagorean Theorem: $\sqrt{1^2+1^2}=\sqrt{2}$ and $\sqrt{2^2-1^2}=\sqrt{3}$. We can use these two special right triangles to make a table of trigonometric ratios for some common angles.

θ in degrees	θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$

EXAMPLE 18 A surveyor located on level ground at a point A is standing 36 m from the base B of a flagpole. The angle of elevation between the ground and the top of the pole is 30°. Find the approximate height h of the flagpole.





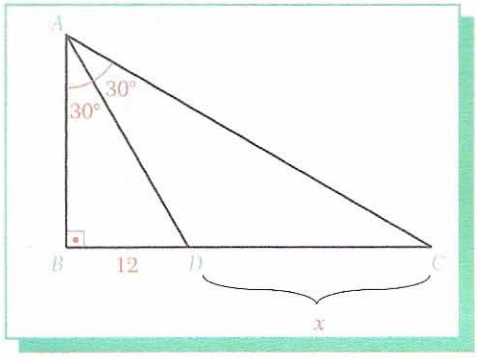


Solution We know that $\tan 30° = \frac{1}{\sqrt{3}}$ from the trigonometric table we have just seen.

Looking at the figure, we also know that $\tan 30° = \frac{opp}{adj} = \frac{h}{36}$.

Therefore $\frac{h}{36} = \frac{1}{\sqrt{3}}$, and so $h = \frac{36}{\sqrt{3}} \cong 20.78$ m.

EXAMPLE 19 In the figure,
$m(\angle BAD) = m(\angle DAC) = 30°$
and $BD = 12$ units.
Find the value of x.

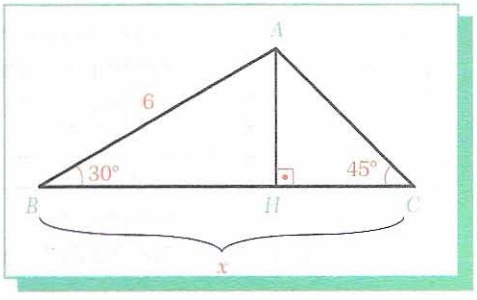


Solution In the right triangle ABD, $\tan 30° = \frac{1}{\sqrt{3}} = \frac{12}{AB}$ so $AB = 12\sqrt{3}$.

In the right triangle ABC, $\tan 60° = \sqrt{3} = \frac{12 + x}{12\sqrt{3}}$ so $12 + x = 12\sqrt{3} \cdot \sqrt{3} = 36$.

So $x = 36 - 12 = 24$ units.

EXAMPLE 20 In the figure,
$m(\angle ABC) = 30°$,
$m(\angle ACB) = 45°$
and $AB = 6$ units.
Find the value of x.



Solution In the right triangle ABH, $\sin 30° = \frac{1}{2} = \frac{AH}{6}$ so $AH = 3$.

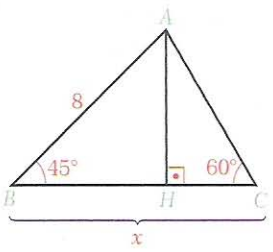
Also, in the same triangle $\cos 30° = \frac{\sqrt{3}}{2} = \frac{BH}{6}$ so $BH = 3\sqrt{3}$. Since angle H is a right angle and angle C measures 45° then in the triangle AHC, $m(\angle A)$ is also 45°. Therefore AHC is an isosceles right triangle.

So $AH = HC = 3$. Since $BC = BH + HC$, we have $BC = x = 3\sqrt{3} + 3$.

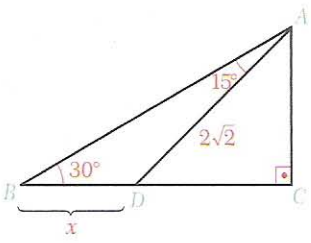
Check Yourself 6

1. Find the length x in each triangle.

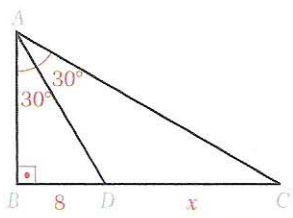
a.



b.



c.



2. Solve for x: $\dfrac{\tan 30° \cdot \csc 60°}{\cos 45° \cdot \sin 60°} = x \cdot \cot 30° \sin 45°$.



Answers

1. a. $4(\sqrt{2} + \dfrac{\sqrt{6}}{3})$ b. $2\sqrt{3} - 2$ c. 16 2. $\dfrac{8}{9}$

B. TRIGONOMETRIC IDENTITIES

The trigonometric ratios are related to each other by equations called **trigonometric identities.**

1. Basic Identities

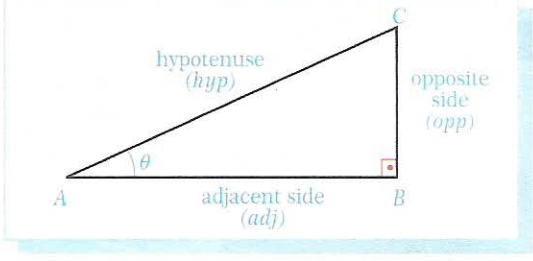
a. Pythagorean identities

Property

Pythagorean identities

For all $\theta \in \mathbb{R}$,

1. $\sin^2\theta + \cos^2\theta = 1$
2. $\tan^2\theta + 1 = \sec^2\theta$
3. $\cot^2\theta + 1 = \csc^2\theta$.



Proof

1. By the Pythagorean Theorem we have $opp^2 + adj^2 = hyp^2$. (1)

Therefore, $\dfrac{opp^2 + adj^2}{hyp^2} = \dfrac{hyp^2}{hyp^2}$ (divide both sides by hyp^2)

$$\frac{opp^2}{hyp^2} + \frac{adj^2}{hyp^2} = \frac{\cancel{hyp^2}}{\cancel{hyp^2}}$$

$$(\frac{opp}{hyp})^2 + (\frac{adj}{hyp})^2 = 1.$$

Since $\sin\theta = \dfrac{opp}{hyp}$ and $\cos\theta = \dfrac{adj}{hyp}$, we have $\sin^2\theta + \cos^2\theta = 1$.

Be careful!
$\sin^2\theta = \sin\theta \cdot \sin\theta$ and $\cos^2\theta = \cos\theta \cdot \cos\theta$, etc. We do not write $\sin\theta^2$ because it is not clear what we mean: $\sin(\theta^2)$ or $(\sin\theta)^2$?

2. Dividing both sides of (1) by adj^2 gives $\frac{opp^2 + adj^2}{adj^2} = \frac{hyp^2}{adj^2}$, i.e.

$\frac{opp^2}{adj^2} + \frac{adj^2}{adj^2} = \frac{hyp^2}{adj^2}$,

$(\frac{opp}{adj})^2 + \frac{\cancel{adj^2}}{\cancel{adj^2}} = (\frac{hyp}{adj})^2$.

Since $\tan\theta = \frac{opp}{adj}$ and $\sec\theta = \frac{hyp}{adj}$ we have $\tan^2\theta + 1 = \sec^2\theta$.

3. Dividing both sides of (1) by opp^2 gives $\frac{opp^2 + adj^2}{opp^2} = \frac{hyp^2}{opp^2}$.

So $\frac{opp^2}{opp^2} + \frac{adj^2}{opp^2} = \frac{hyp^2}{opp^2}$,

$\frac{\cancel{opp^2}}{\cancel{opp^2}} + (\frac{adj}{opp})^2 = (\frac{hyp}{opp})^2$.

Since $\cot\theta = \frac{adj}{opp}$ and $\csc\theta = \frac{hyp}{opp}$ we have $1 + \cot^2\theta = \csc^2\theta$, i.e. $\cot^2\theta + 1 = \csc^2\theta$.

b. Tangent and cotangent identities

Property

tangent and cotangent identities

1. $\tan\theta = \frac{\sin\theta}{\cos\theta}$
2. $\cot\theta = \frac{\cos\theta}{\sin\theta}$
3. $\tan\theta \cdot \cot\theta = 1$

Proof

1. We know that $\sin\theta = \frac{opp}{hyp}$ and $\cos\theta = \frac{adj}{hyp}$, so $\frac{\sin\theta}{\cos\theta} = \frac{\frac{opp}{\cancel{hyp}}}{\frac{adj}{\cancel{hyp}}} = \frac{opp}{adj}$.

So $\frac{\sin\theta}{\cos\theta} = \tan\theta$.

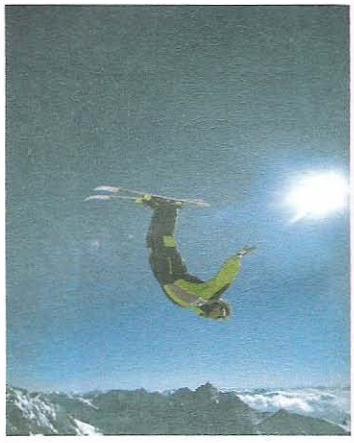
2. Similarly, $\frac{\cos\theta}{\sin\theta} = \frac{\frac{adj}{\cancel{hyp}}}{\frac{opp}{\cancel{hyp}}} = \frac{adj}{opp}$, i.e. $\frac{\cos\theta}{\sin\theta} = \cot\theta$.

3. Consequently, $\frac{\cancel{\sin\theta}}{\cancel{\cos\theta}} \cdot \frac{\cancel{\cos\theta}}{\cancel{\sin\theta}} = 1$. So $\tan\theta \cdot \cot\theta = 1$.

c. Reciprocal identities

Property reciprocal identities

1. $\csc\theta = \dfrac{1}{\sin\theta}$
2. $\sec\theta = \dfrac{1}{\cos\theta}$



Proof

1. We know that $\sin\theta = \dfrac{opp}{hyp}$, so $\dfrac{1}{\sin\theta} = \dfrac{1}{\frac{opp}{hyp}} = \dfrac{hyp}{opp}$.

 Since $\csc\theta = \dfrac{hyp}{opp}$, $\csc\theta = \dfrac{1}{\sin\theta}$.

2. Similarly, $\cos\theta = \dfrac{adj}{hyp}$ so $\dfrac{1}{\cos\theta} = \dfrac{1}{\frac{adj}{hyp}} = \dfrac{hyp}{adj}$.

 Since $\sec\theta = \dfrac{hyp}{adj}$, $\sec\theta = \dfrac{1}{\cos\theta}$.

Remember!

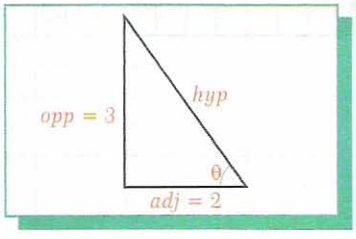
$\csc\theta = \dfrac{1}{\sin\theta}$

$\sec\theta = \dfrac{1}{\cos\theta}$

(the first letters of reciprocal ratios are opposite:

$c = \dfrac{1}{s}, s = \dfrac{1}{c}$)

EXAMPLE 21 Verify the eight trigonometric identities using the right triangle in the figure.



Solution First we need to calculate the length of the hypotenuse.

By the Pythagorean Theorem, $hyp^2 = 2^2 + 3^2$

$hyp^2 = 4 + 9$

$hyp^2 = 13$

$hyp = \sqrt{13}$.

Before verifying the identities, let us write the six trigonometric ratios for the given right triangle:

$\sin\theta = \dfrac{3}{\sqrt{13}}$, $\cos\theta = \dfrac{2}{\sqrt{13}}$, $\tan\theta = \dfrac{3}{2}$, $\cot\theta = \dfrac{2}{3}$, $\csc\theta = \dfrac{\sqrt{13}}{3}$, $\sec\theta = \dfrac{\sqrt{13}}{2}$.

Now we can verify the identities.

$\tan\theta \cdot \cot\theta = \dfrac{\cancel{3}}{\cancel{2}} \cdot \dfrac{\cancel{2}}{\cancel{3}} = 1$

$\dfrac{1}{\sin\theta} = \dfrac{1}{\frac{3}{\sqrt{13}}} = \dfrac{\sqrt{13}}{3} = \csc\theta$

$\dfrac{1}{\cos\theta} = \dfrac{1}{\frac{2}{\sqrt{13}}} = \dfrac{\sqrt{13}}{2} = \sec\theta$

We know that $\tan\theta = \frac{3}{2}$ and $\sec\theta = \frac{\sqrt{13}}{2}$, so by substitution,

$$(\frac{3}{2})^2 + 1 = (\frac{\sqrt{13}}{2})^2$$

$$\frac{9}{4} + 1 = \frac{13}{4}$$

$\frac{13}{4} = \frac{13}{4}$. Therefore, $\tan^2\theta + 1 = \sec^2\theta$.

We know that $\cot\theta = \frac{2}{3}$ and $\csc\theta = \frac{\sqrt{13}}{3}$, so by substitution,

$$(\frac{2}{3})^2 + 1 = (\frac{\sqrt{13}}{3})^2$$

$$\frac{4}{9} + 1 = \frac{13}{9}$$

$\frac{13}{9} = \frac{13}{9}$. Therefore, $\cot^2\theta + 1 = \csc^2\theta$,

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{3}{\sqrt{13}}}{\frac{2}{\sqrt{13}}} = \frac{3}{2} = \tan\theta, \quad \frac{\cos\theta}{\sin\theta} = \frac{\frac{2}{\sqrt{13}}}{\frac{3}{\sqrt{13}}} = \frac{2}{3} = \cot\theta,$$

$$\sin^2\theta + \cos^2\theta = (\frac{3}{\sqrt{13}})^2 + (\frac{2}{\sqrt{13}})^2 = \frac{9}{13} + \frac{4}{13} = \frac{13}{13} = 1.$$

Check Yourself 7

1. Verify the eight trigonometric identities for the acute angle θ in a right triangle if $\sin\theta = \frac{2}{5}$.
2. Verify the eight trigonometric identities for a right triangle with sides of length 7, 24 and 25 units.
3. Let α be an acute angle in a right triangle such that $\sin\alpha = \frac{4}{5}$. Evaluate $\cos\alpha \cdot (\tan\alpha + \cot\alpha)$.
4. Evaluate $\dfrac{\cos^2\frac{\pi}{5} + \sin^2\frac{21\pi}{5} + 1}{2 - \tan\frac{\pi}{7} \cdot \cot\frac{\pi}{7}}$.

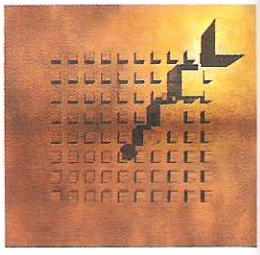
Answers

3. $\frac{5}{4}$ 4. 2

2. Simplifying Trigonometric Expressions

In the previous section we studied the eight most common trigonometric identities. These identities are useful when we are simplifying trigonometric expressions. Let us look at some examples.

EXAMPLE 22 Simplify $\cos x \cdot \tan x$.



Solution We can use the identity $\tan x = \dfrac{\sin x}{\cos x}$:

$$\cos x \cdot \tan x = \cos x \cdot \frac{\sin x}{\cos x} \qquad \text{(substitute)}$$

$$= \cancel{\cos x} \cdot \frac{\sin x}{\cancel{\cos x}} \qquad \text{(cancel)}$$

$$= \sin x.$$

So $\cos x \cdot \tan x = \sin x$.

EXAMPLE 23 a. Simplify $\tan x \cdot \cos x \cdot \csc x$.

b. Simplify $\cos^3 x + \sin^2 x \cdot \cos x$.



Solution a. We know $\tan x = \dfrac{\sin x}{\cos x}$ and $\csc x = \dfrac{1}{\sin x}$. Hence,

$$\tan x \cdot \cos x \cdot \csc x = \frac{\sin x}{\cos x} \cdot \cos x \cdot \frac{1}{\sin x} \qquad \text{(substitute)}$$

$$= \frac{\cancel{\sin x}}{\cancel{\cos x}} \cdot \cancel{\cos x} \cdot \frac{1}{\cancel{\sin x}} = 1. \qquad \text{(cancel)}$$

So $\tan x \cdot \cos x \cdot \csc x = 1$.

b. Since $\cos x$ is the common factor in both terms of the expression, let us factorize the expression:

$$\cos^3 x + \sin^2 x \cdot \cos x = \cos x \cdot (\cos^2 x + \sin^2 x) \qquad \text{(factorize)}$$

$$= \cos x \cdot 1 = \cos x \qquad \text{(using } \cos^2 x + \sin^2 x = 1\text{)}$$

So $\cos^3 x + \sin^2 x \cdot \cos x = \cos x$.

EXAMPLE 24 Simplify $\dfrac{\sec x - \cos x}{\tan x}$.



Solution We know $\tan x = \dfrac{\sin x}{\cos x}$ and $\sec x = \dfrac{1}{\cos x}$. Hence,

$$\frac{\sec x - \cos x}{\tan x} = \frac{\dfrac{1}{\cos x} - \cos x}{\dfrac{\sin x}{\cos x}}$$ (by substitution)

$$= \frac{\dfrac{1}{\cos x} - \dfrac{\cos^2 x}{\cos x}}{\dfrac{\sin x}{\cos x}}$$ (equalize the denominators)

$$= \frac{\dfrac{1-\cos^2 x}{\cos x}}{\dfrac{\sin x}{\cos x}}$$ (simplify the numerator)

$$= \frac{\dfrac{1-\cos^2 x}{\cancel{\cos x}}}{\dfrac{\sin x}{\cancel{\cos x}}}$$ (cancel the common divisor)

$$= \frac{\sin^2 x}{\sin x}$$ (using $\sin^2 x + \cos^2 x = 1$)

$$= \sin x.$$ (by cancellation)

As a result, $\dfrac{\sec x - \cos x}{\tan x} = \sin x$.

EXAMPLE 25 Simplify $\dfrac{2 + \tan^2 x}{\sec^2 x} - 1$

Solution We know $\tan x = \dfrac{\sin x}{\cos x}$ and $\sec x = \dfrac{1}{\cos x}$. Hence,



$$\frac{2+\tan^2 x}{\sec^2 x}-1=\frac{2+(\frac{\sin x}{\cos x})^2}{(\frac{1}{\cos x})^2}-1 \qquad \text{(by substitution)}$$

$$=\frac{2+\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}}-1$$

$$=\frac{\frac{2\cdot\cos^2 x}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}}-1 \qquad \text{(equalize the denominators in the numerator)}$$

$$=\frac{\frac{2\cdot\cos^2 x+\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}}-1 \qquad \text{(simplify the numerator)}$$

$$=\frac{\frac{2\cdot\cos^2 x+\sin^2 x}{\cancel{\cos^2 x}}}{\frac{1}{\cancel{\cos^2 x}}}-1 \qquad \text{(cancel the common divisor)}$$

$$=\cos^2 x+\cos^2 x+\sin^2 x-1 \qquad (2\cdot\cos^2 x=\cos^2 x+\cos^2 x)$$

$$=\cos^2 x+\cancel{1}-\cancel{1} \qquad (\text{using } \cos^2 x+\sin^2 x=1)$$

$$=\cos^2 x+1-1 \qquad \text{(by cancellation)}$$

$$=\cos^2 x.$$

As a result, $\frac{2+\tan^2 x}{\sec^2 x}-1=\cos^2 x.$

Check Yourself 8

Simplify the expressions.

1. $\cos x\cdot\tan x$
2. $\frac{1+\csc x}{\cos x+\cot x}$
3. $\frac{1}{1-\sin x}+\frac{1}{1+\sin x}$

Answers

1. $\sin x$
2. $\sec x$
3. $2\cdot\sec^2 x$

3. Verifying Trigonometric Identities

In the previous section we learned how to write a trigonometric expression in an alternative (simpler) form using the eight basic identities. This means that we can derive other identities using the eight basic identities. In this section we will learn how to verify a given trigonometric identity.

To verify an identity, we try to show that one side of the identity is the same as the other side. We take either the left-hand or right-hand side of the identity and do algebraic operations to obtain the other side. Generally, it is easier to begin working with the more complex side of the identity.

Let us look at some examples.

EXAMPLE 26 Verify the identity $\sin x \cdot \cot x = \cos x$.

Solution We can begin with either the left-hand side or the right-hand side. In this example we will show both approaches.

Working on the left-hand side:

$$\sin x \cdot \cot x = \sin x \cdot \frac{\cos x}{\sin x} \qquad \text{(by substitution)}$$

$$= \cancel{\sin x} \cdot \frac{\cos x}{\cancel{\sin x}} = \cos x. \qquad \text{(simplify)}$$

We have obtained the right-hand side and the verification is complete.

Working on the right-hand side:

$$\cos x = \cos x \cdot 1 \qquad \text{(by substitution)}$$

$$= \cos x \cdot \tan x \cdot \cot x \qquad (\tan x \cdot \cot x = 1)$$

$$= \cos x \cdot \frac{\sin x}{\cos x} \cdot \cot x \qquad \text{(by substitution)}$$

$$= \cancel{\cos x} \cdot \frac{\sin x}{\cancel{\cos x}} \cdot \cot x = \sin x \cdot \cot x. \qquad \text{(simplify)}$$

We have obtained the left-hand side and the verification is complete.

EXAMPLE 27 Verify the identity $\csc x = \cos x \cdot (\tan x + \cot x)$.

Solution Since the right-hand side is more complex than the left-hand side, let us try to transform the right-hand side into the left-hand side.

We know $\tan x = \dfrac{\sin x}{\cos x}$ and $\cot x = \dfrac{\cos x}{\sin x}$, so

$$\cos x \cdot (\tan x + \cot x) = \cos x \cdot \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right) \quad \text{(by substitution)}$$

$$= \cos x \cdot \left(\frac{\sin^2 x + \cos^2 x}{\cos x \cdot \sin x}\right) \quad \text{(equalize the denominators)}$$

$$= \cancel{\cos x} \cdot \left(\frac{1}{\cancel{\cos x} \cdot \sin x}\right) \quad \text{(cancel)}$$

$$= \frac{1}{\sin x} = \csc x.$$

We have obtained the other side of the identity and the verification is complete.

EXAMPLE 28 Verify the identity $\dfrac{(\sin x + \cos x)^2}{\sin x \cdot \cos x} = 2 + \sec x \cdot \csc x.$

Solution Let us begin with the left-hand side as it is more complex.

$$\frac{(\sin x + \cos x)^2}{\sin x \cdot \cos x} = \frac{\sin^2 x + (2 \cdot \sin x \cdot \cos x) + \cos^2 x}{\sin x \cdot \cos x} \quad \text{(expand the numerator)}$$

$$= \frac{(2 \cdot \sin x \cdot \cos x) + 1}{\sin x \cdot \cos x} \quad (\cos^2 x + \sin^2 x = 1)$$

$$= \frac{2 \cdot \sin x \cdot \cos x}{\sin x \cdot \cos x} + \frac{1}{\sin x \cdot \cos x} \quad \text{(separate the fractions)}$$

$$= 2 + \csc x \cdot \sec x \quad \text{(simplify)}$$

We have obtained the right-hand side of the identity and the verification is complete.

EXAMPLE 29 Verify the identity $\dfrac{\tan x}{\csc x} = \sec x - \cos x.$

Solution Let us work on the left-hand side.

$$\frac{\tan x}{\csc x} = \frac{\dfrac{\sin x}{\cos x}}{\dfrac{1}{\sin x}} \quad \text{(by substitution)}$$

$$= \frac{\sin x}{\cos x} \cdot \frac{\sin x}{1} \quad \text{(invert the denominator and multiply)}$$

$$= \frac{\sin^2 x}{\cos x} \qquad \text{(multiply)}$$

$$= \frac{1-\cos^2 x}{\cos x} \qquad (\cos^2 x + \sin^2 x = 1)$$

$$= \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x} \qquad \text{(separate the fractions)}$$

$$= \sec x - \cos x \qquad \text{(substitute and simplify)}$$

We have obtained the right-hand side of the identity and so the verification is complete.

EXAMPLE 30 Verify the identity $\dfrac{\cos x}{1-\sin x} = \dfrac{1+\sin x}{\cos x}$.

Solution Begin with the left-hand side.

$$\frac{\cos x}{1-\sin x} = \frac{\cos x}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x} \qquad \text{(multiply by 1)}$$

$$= \frac{\cos x \cdot (1+\sin x)}{1-\sin^2 x} \qquad \text{(write the product)}$$

$$= \frac{\cos x \cdot (1+\sin x)}{\cos^2 x} \qquad (\cos^2 x + \sin^2 x = 1)$$

$$= \frac{1+\sin x}{\cos x} \qquad \text{(cancel the common factor)}$$

This is the right-hand side of the identity, so the verification is complete.

Check Yourself 9

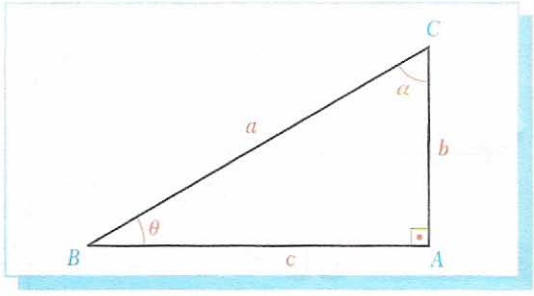
Verify the identities.

1. $\sec x - \cos x = \sin x \cdot \tan x$
2. $\dfrac{\cos x}{\sec x \cdot \sin x} = \csc x - \sin x$
3. $\dfrac{1+\sin x}{\cos x} + \dfrac{\cos x}{1+\sin x} = 2\sec x$

4. Cofunctions

We have studied the trigonometric functions of certain angles and the trigonmetric ratios between the sides and angles of a right triangle. In this section we will look at the relation between the trigonometric ratios of complementary angles.

Recall that complementary angles are angles whose sum is 90°, i.e. $\frac{\pi^R}{2}$. Consider the right triangle ABC with acute angles θ and α shown in the figure. θ and α are complementary angles.



We can also write $\sin\theta = \cos\alpha = \frac{b}{a}$. In other words, the sine of θ and the cosine of its complement are equal. We say that sine and cosine are **cofunctions**. Looking at the triangle we can also write $\tan\theta = \cot\alpha = \frac{b}{c}$ (so tangent and cotangent are cofunctions) and $\sec\theta = \csc\alpha = \frac{a}{c}$ (i.e. secant and cosecant are cofunctions).

In other words, for $\theta + \alpha = 90°$, $\theta = 90° - \alpha$ we have

$$\sin\theta = \sin(90° - \alpha) = \cos\alpha$$

$$\tan\theta = \tan(90° - \alpha) = \cot\alpha$$

$$\sec\theta = \sec(90° - \alpha) = \csc\alpha.$$

For example,

$$\sin 43° = \cos(90° - 43°) = \cos 47°,$$

$$\cos 26° = \sin 63°,$$

$$\tan 3° = \cot 87°,$$

$$\sec 18° = \csc 72°,$$

$$\sin\frac{\pi}{5} = \cos(\frac{\pi}{2} - \frac{\pi}{5}) = \cos\frac{3\pi}{10} \text{ and}$$

$$\cos\frac{3\pi}{8} = \sin(\frac{\pi}{2} - \frac{3\pi}{8}) = \sin\frac{\pi}{8}, \text{ etc.}$$

EXAMPLE 31 Evaluate each expression.

a. $\tan 1° \cdot \tan 2° \cdot \tan 3° \cdot \ldots \cdot \tan 88° \cdot \tan 89°$

b. $\sin^2\frac{\pi}{7} + [\tan\frac{7\pi}{18} \cdot \tan\frac{\pi}{9}] + \sin^2\frac{5\pi}{14}$

Solution a. The angles in each pair (89°, 1°), (88°, 2°), ..., (46°, 44°) are complementary.

Because tangent and cotangent are cofunctions,

$\tan 89° = \cot 1°, \tan 88° = \cot 2°, \ldots, \tan 46° = \cot 44°.$

So $\tan 1° \cdot \tan 2° \cdot \tan 3° \cdot \ldots \cdot \tan 88° \cdot \tan 89°$

$= \tan 1° \cdot \tan 2° \cdot \tan 3° \cdot \ldots \cdot \tan 44° \cdot \tan 45° \cdot \cot 44° \cdot \ldots \cdot \cot 2° \cdot \cot 1°$

$= 1 \cdot 1 \cdot 1 \cdot \ldots \cdot \tan 45° \cdot \ldots \cdot 1 \cdot 1 = 1.$

b. The complement of $\frac{\pi}{7}$ is $\frac{5\pi}{14}$ since $\frac{\pi}{7} + \frac{5\pi}{14} = \frac{\pi}{2}$.

Similarly, the complement of

$\frac{7\pi}{18}$ is $\frac{\pi}{9}$ since $\frac{7\pi}{18} + \frac{\pi}{9} = \frac{\pi}{2}$.

So $\sin\frac{5\pi}{14} = \cos\frac{\pi}{7}$ and $\tan\frac{7\pi}{18} = \cot\frac{\pi}{9}$ since these are cofunctions.

So $\sin^2\frac{\pi}{7} + \tan\frac{7\pi}{18} \cdot \tan\frac{\pi}{9} + \sin^2\frac{5\pi}{14} = \sin^2\frac{\pi}{7} + \left[\cot\frac{\pi}{9} \cdot \tan\frac{\pi}{9}\right] + \cos^2\frac{\pi}{7}$

$= \underbrace{\sin^2\frac{\pi}{7} + \cos^2\frac{\pi}{7}}_{1} + \underbrace{\cot\frac{\pi}{9} \cdot \tan\frac{\pi}{9}}_{1}$

$= 1 + 1$

$= 2.$

Check Yourself 10

1. Write the cofunction of each function.

 a. $\tan 15°$ b. $\cos 36°$ c. $\sec 77°$

 d. $\sin\frac{\pi}{12}$ e. $\cot\frac{2\pi}{5}$ f. $\tan\frac{2\pi}{7}$

2. Evaluate each expression.

 a. $\tan 5° \cdot \tan 25° \cdot \tan 45° \cdot \tan 65° \cdot \tan 85°$

 b. $\tan\frac{\pi}{7} \cdot \tan\frac{5\pi}{14} - \cos^2 27° - \cos^2 63°$

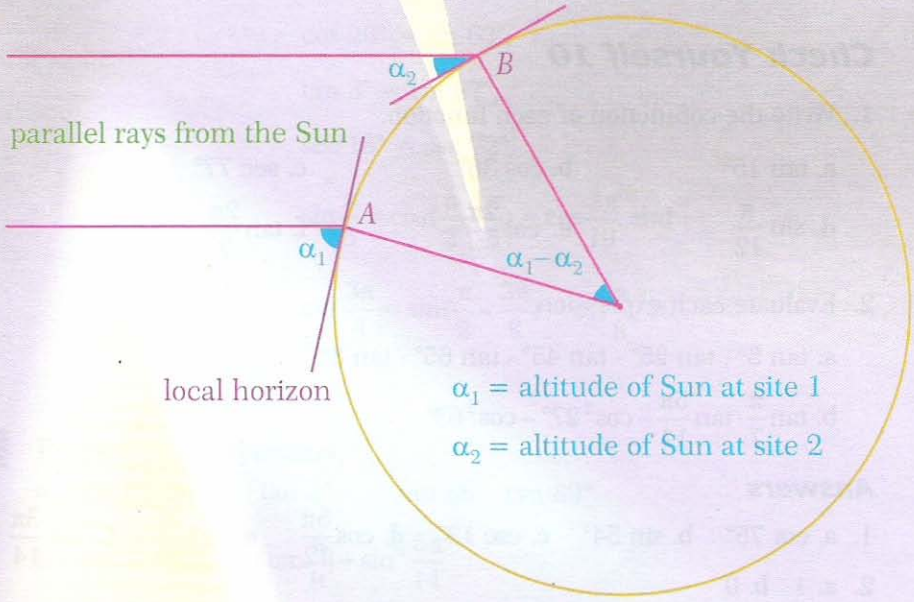
Answers

1. a. $\cot 75°$ b. $\sin 54°$ c. $\csc 13°$ d. $\cos\frac{5\pi}{12}$ e. $\tan\frac{\pi}{10}$ f. $\cot\frac{3\pi}{14}$
2. a. 1 b. 0

CALCULATING THE EARTH'S CIRCUMFERENC

Eratosthenes was a famous mathematician and the head of the famous library in Alexandria, Egypt. In 240 BC he calculated the Earth's circumference using trigonometry and his knowledge of the angle of elevation of the Sun at the summer solstice in the Egyptian cities of Alexandria and Syene (now called Aswan). Eratosthenes' calculation was based on the assumptions that the Earth is a sphere and that the sun is so far away that we can consider its rays to be parallel.

Eratosthenes compared observations made in Alexandria, where the noontime Sun at the summer solstice was 7° away from straight overhead (the zenith), to observations in Syene in southern Egypt, where the Sun was exactly at its zenith. The distance between the cities was known to be about 5000 stadia, roughly equal to 800 km (the stadion, plural stadia, was an old unit of measurement such that 1 stadia ≅ 160 m). Therefore, Eratosthenes calculated the entire 360° circle of the Earth to be (360/7) · 5000 stadia, which is about 260,000 stadia, or 41,000 km.



circumference of the Earth = arc length $\widehat{AB}$ · (360° / ($\alpha - \alpha_1$))

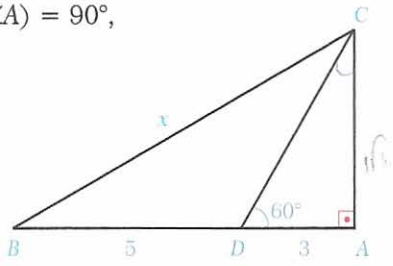
EXERCISES 2

A. Trigonometric Ratios

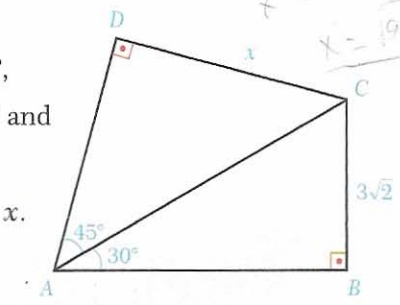
1. $\tan x = \dfrac{1}{\sqrt{17}}$ is given. Evaluate each ratio, given that x is in the first quadrant.

 a. $\cot x$ b. $\sin x$ c. $\cos x$

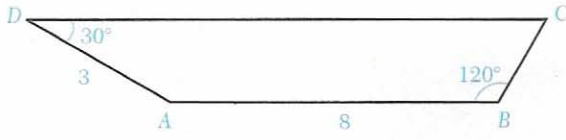
2. In the figure, $m(\angle A) = 90°$, $AD = 3$, $DB = 5$ and $m(\angle ADC) = 60°$. Calculate $BC = x$.



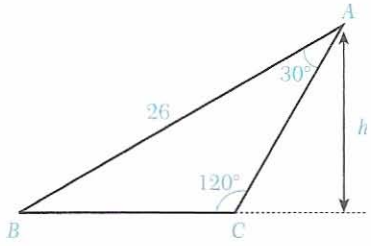
3. In the figure, $m(\angle DAC) = 45°$, $m(\angle BAC) = 30°$ and $BC = 3\sqrt{2}$. Calculate $DC = x$.



4. In the figure, $DC \parallel AB$, $m(\angle D) = 30°$, $m(\angle B) = 120°$, $AD = 3$ and $AB = 8$. Calculate DC.

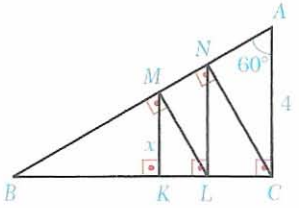


5. In the figure, $m(\angle C) = 120°$, $m(\angle A) = 30°$ and $AB = 26$. Calculate the height h.

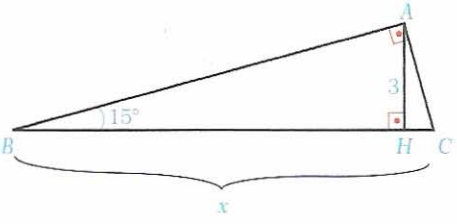


6. Calculate the length x in each figure.

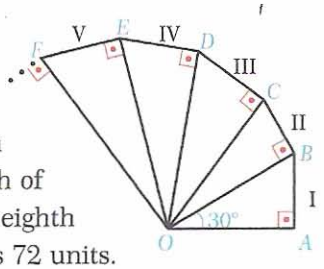
 a.



 b.



7. The figure shows a series of 30°-60°-90° right triangles, increasing in size from right to left. The length of the hypotenuse of the eighth triangle in the series is 72 units. Calculate AO.



B. Trigonometric Identities

8. Simplify the expressions.

 a. $\csc x \cdot \tan x$ b. $\sec^2 x - \tan^2 x$

 c. $\tan x + \cot x$ d. $\dfrac{1+\sin x}{1+\csc x}$

 e. $\dfrac{\tan x}{\sec x}$ f. $\dfrac{\cot x - 1}{1 - \tan x}$

9. Simplify the expressions.

a. $(\sec x - \tan x)^2 (1 + \sin x)$

b. $\dfrac{\cos x}{\sec x + \tan x}$

c. $\sin^4 x - \cos^4 x + \cos^2 x$

d. $\dfrac{\sin x}{1-\cos x} - \csc x$

e. $\dfrac{1}{\sec^2 x} + \dfrac{1}{\csc^2 x}$

f. $(\tan x + \sin x)^2 + (1 + \cos x)^2$

10. Verify the identities.

a. $(1 - \cos x)(1 - \cos x) = \sin^2 x$

b. $\dfrac{\cos x}{\sec x} + \dfrac{\sin x}{\csc x} = 1$

c. $\dfrac{(\sin x + \cos x)^2}{\sin^2 x - \cos^2 x} = \dfrac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2}$

d. $\dfrac{1-\sin x}{1+\sin x} = (\sec x - \tan x)^2$

e. $\csc x - \sin x = \cos x \cot x$

f. $(\cot x - \csc x)(\cos x + 1) = -\sin x$

11. Verify the identities.

a. $\dfrac{1+\tan x}{1-\tan x} = \dfrac{\cos x + \sin x}{\cos x - \sin x}$

b. $\dfrac{\cos x}{1-\sin x} = \dfrac{\sin x - \cos x}{\cos x - \cot x}$

c. $\dfrac{1}{1-\sin x} - \dfrac{1}{1+\sin x} = 2\sec x \tan x$

d. $\dfrac{\tan x - \cot x}{\tan^2 x - \cot^2 x} = \sin x \cos x$

e. $\dfrac{1+\sin x}{1-\sin x} = (\tan x + \sec x)^2$

f. $\dfrac{1+\cos x}{1-\cos x} = (\cot x + \csc x)^2$

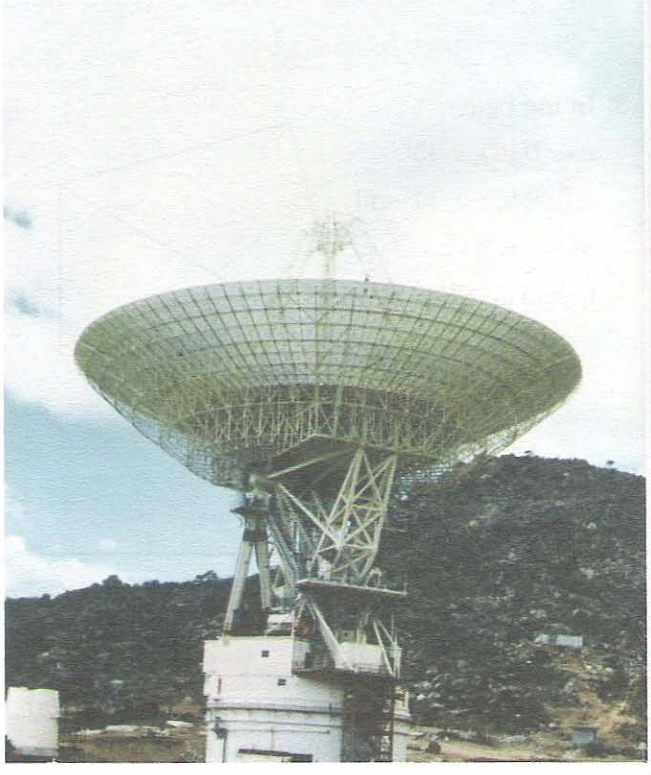
12. Evaluate the expressions.

a. $\dfrac{\tan\dfrac{2\pi}{7} \cdot \tan\dfrac{3\pi}{14} - \sin^2\dfrac{\pi}{10}}{1 - \cos^2\dfrac{2\pi}{5}}$

b. $\dfrac{\tan 25^\circ \cdot \tan 65^\circ}{\sin^2 25^\circ + \sin^2 65^\circ} + 2$

13. Evaluate

$$\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8} + \cos^2\frac{\pi}{12} + \cos^2\frac{5\pi}{12}.$$



3 TRIGONOMETRIC FUNCTIONS OF REAL NUMBERS

A. TRIGONOMETRIC FUNCTIONS

Up to now we have defined the trigonometric ratios of acute angles in a right triangle. However, we can in fact calculate trigonometric ratios for any angle. In this section we will extend our knowledge of trigonometric ratios to cover all angles. To do this, we will study the trigonometric functions in the context of the unit circle. From now on in this module, we will use the terms **trigonometric ratio, trigonometric value** and **trigonometric function value** interchangeably.

The trigonometric functions are also called **circular functions** since they can be defined on the unit circle.

1. The Sine Function

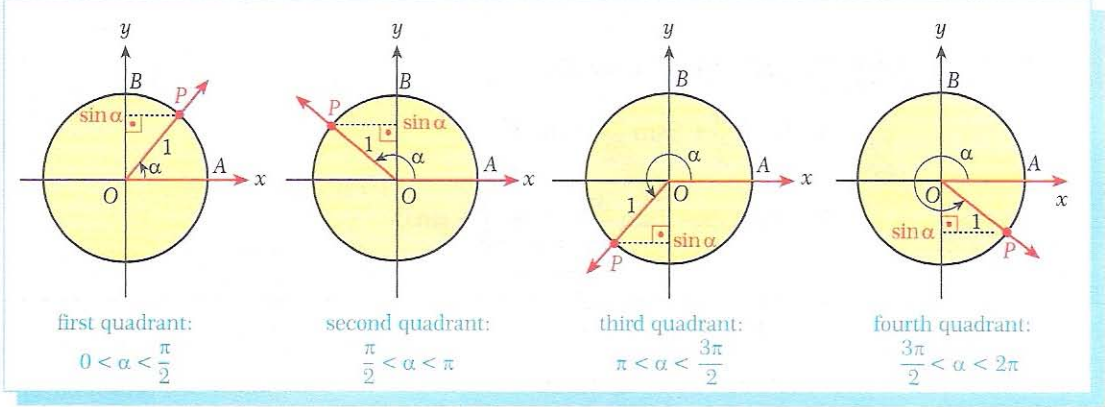
Definition

sine of an angle, sine function

Let $[OP]$ be the terminal side of an angle α in standard position such that P lies on the unit circle. Then the ordinate (y-coordinate) of the point P on the unit circle is called the **sine of angle α**. It is denoted by **sin α**. The function which matches a real number α to the real number $\sin \alpha$ is called the **sine function.**

Since the sine value of α is the ordinate of the point P, the y-axis can also be called the **sine axis**.

The figures below show how the value of $\sin \alpha$ changes as the point P moves round the unit circle.



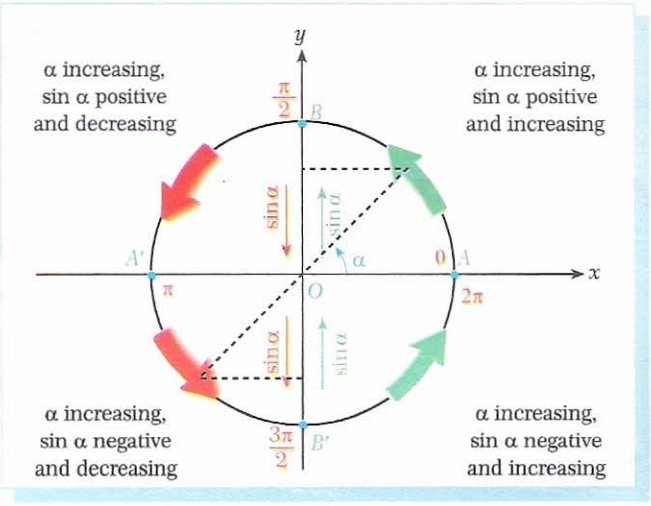
As we can see in the figures:

if $0 < \alpha < \frac{\pi}{2}$ then $0 < \sin\alpha < 1$;

if $\frac{\pi}{2} < \alpha < \pi$ then $0 < \sin\alpha < 1$;

if $\pi < \alpha < \frac{3\pi}{2}$ then $-1 < \sin\alpha < 0$;

if $\frac{3\pi}{2} < \alpha < 2\pi$ then $-1 < \sin\alpha < 0$.





Since sin α is the ordinate of the point P, we can also easily observe how it increases and decreases as P moves round the circle.

Conclusion

1. For all values of α the sine function takes values between –1 and 1, i.e. $-1 \leq \sin\alpha \leq 1$.
2. As α increases, the sine function **increases** in the first and fourth quadrants and **decreases** in the second and third quadrants.
3. The sine function is **positive** in the first and second quadrants and **negative** in the third and fourth quadrants.

EXAMPLE 32 Simplify the expressions.

a. $3 \cdot \sin\frac{\pi}{2} + 5\sin\frac{3\pi}{2} - \sin\pi$

b. $-4\sin 2\pi - 2\sin\frac{3\pi}{2} + 4\sin\frac{\pi}{2} + \sin 0$

Solution a. $3 \cdot 1 + 5 \cdot (-1) - 0 = -2$

b. $-4 \cdot 0 - 2 \cdot (-1) + 4 \cdot 1 + 0 = 6$

2. The Cosine Function

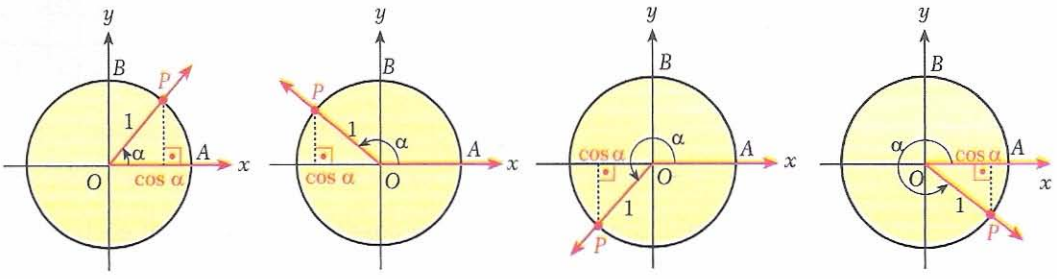
Definition

cosine of an angle, cosine function

Let $[OP]$ be the terminal side of an angle α in standard position such that P lies on the unit circle. Then the abscissa (x-coordinate) of the point P on the unit circle is called the **cosine of angle α**. It is denoted by **cos α**. The function which matches a real number α to the real number cos α is called the **cosine function.**

The figures below show how the value of cos α changes as the point P moves round the unit circle.

Since the cosine value of α is the abscissa of the point P, the x-axis can also be called the **cosine axis.**



first quadrant: $0 < \alpha < \frac{\pi}{2}$

second quadrant: $\frac{\pi}{2} < \alpha < \pi$

third quadrant: $\pi < \alpha < \frac{3\pi}{2}$

fourth quadrant: $\frac{3\pi}{2} < \alpha < 2\pi$

As we can see in the figures:

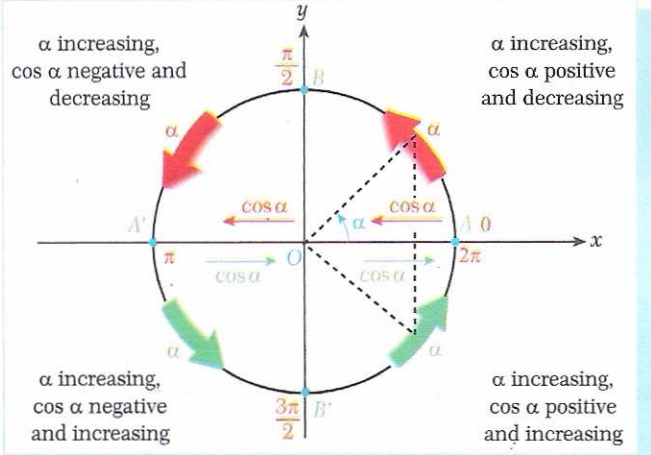
if $0 < \alpha < \frac{\pi}{2}$ then $0 < \cos \alpha < 1$;

if $\frac{\pi}{2} < \alpha < \pi$ then $-1 < \cos \alpha < 0$;

if $\pi < \alpha < \frac{3\pi}{2}$ then $-1 < \cos \alpha < 0$;

if $\frac{3\pi}{2} < \alpha < 2\pi$ then $0 < \cos \alpha < 1$.

Since cos α is the abscissa of the point P, we can also easily observe how it increases and decreases as P moves round the circle.



Conclusion

1. For all values of α, the cosine function takes values between –1 and 1, i.e. $-1 \leq \cos \alpha \leq 1$.
2. As α increases, the cosine function **decreases** in the first and second quadrants and **increases** in the third and fourth quadrants.
3. The cosine function is **positive** in the first and fourth quadrants and **negative** in the second and third quadrants.

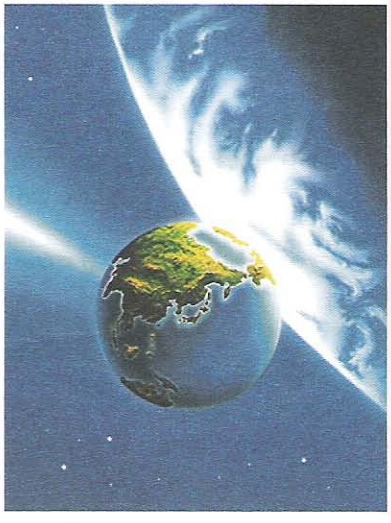
Notice that substituting the values $y = \sin \theta$ and $x = \cos \theta$ in the unit circle equation $x^2 + y^2 = 1$ gives us the relation $\cos^2 \theta + \sin^2 \theta = 1$.



EXAMPLE 33 Find the minimum and maximum possible values of A in each expression.

a. $A = 3\cos x - 1$

b. $A = 2 - 4\sin x - 1$



Solution

a. We know that $-1 \leq \cos x \leq 1$, so

$$-3 \leq 3\cos x \leq 3$$

$$-3 - 1 \leq 3\cos x - 1 \leq 3 - 1$$

$$-4 \leq 3\cos x - 1 \leq 2.$$

So $\min(A) = -4$ and $\max(A) = 2$.

b. We know that $-1 \leq \sin x \leq 1$, so

$$-4 \leq 4\sin x \leq 4$$

$$+4 \geq -4\sin x \geq -4.$$

We can rearrange this to get

$$-4 \leq -4\sin x \leq 4$$

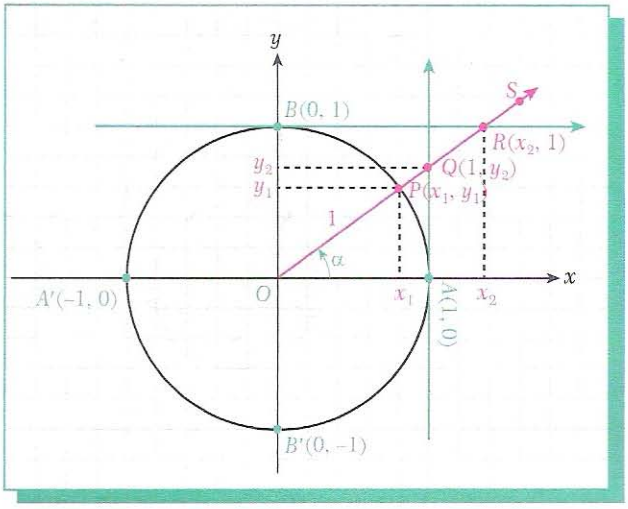
$$-4 + 2 \leq 2 - 4\sin x \leq 4 + 2$$

$$-2 \leq (2 - 4\sin x) \leq 6.$$

So $\min(A) = -2$ and $\max(A) = 6$.

3. The Tangent Function

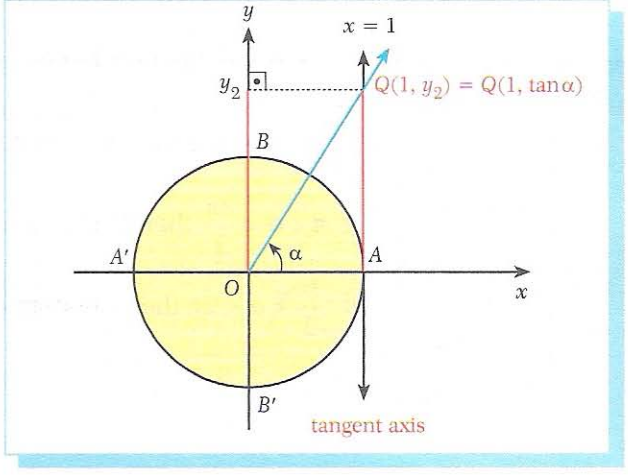
To define the tangent and cotangent functions, we begin by constructing a unit circle. Then we draw the lines $x = 1$ and $y = 1$ tangent to the circle at the points $A(1,0)$ and $B(0, 1)$ respectively. The terminal side of the angle α intersects the lines $x = 1$ and $y = 1$ at $Q(1, y_2)$ and $R(x_2, 1)$ respectively.



Definition

tangent of an angle, tangent function

Let $[OQ]$ be the terminal side of an angle α as shown in the figure such that point Q lies on the line $x = 1$. Then the ordinate (y-coordinate) of point Q on the line $x = 1$ is called the **tangent of angle α**. It is denoted by **tan α**. The function which matches a real number α to the real number tan α is called the **tangent function.**

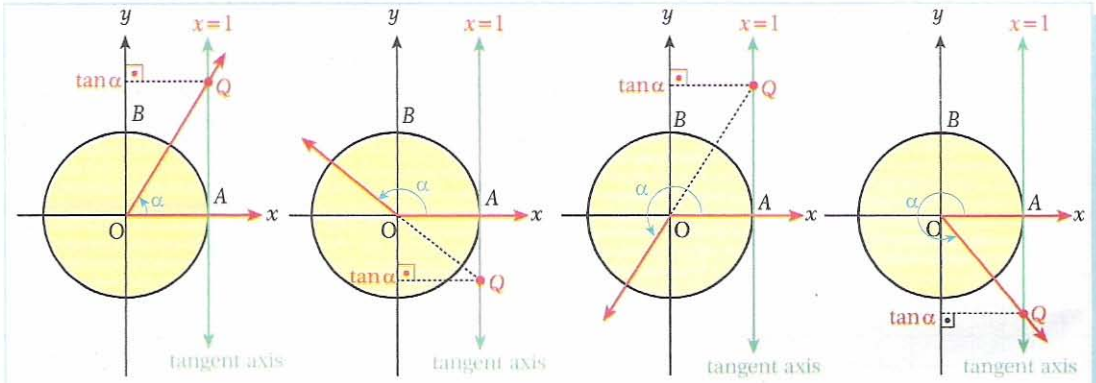


Since the tangent value of α is the ordinate of the point $Q(1, y)$, the line $x = 1$ can also be called the tangent axis.

Note

The real numbers $\frac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$ correspond to the points $B(0, 1)$ and $B'(0, -1)$ on the circle. At these points the terminal side of the angle α coincides with the positive or negative y-axis. Since the y-axis is parallel to the line $x = 1$, the intersection point Q does not exist and so at these points the tangent function is not defined.

The figures below show how the value of tan α changes as α moves round the circle. Notice that we extend the terminal side of the angle in the second and third quadrants to find its intersection with the tangent line $x = 1$.



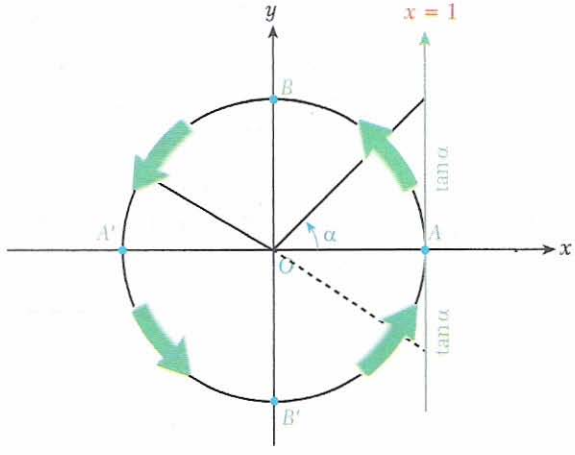
As we can see in the figures:

if $0 < \alpha < \frac{\pi}{2}$ then $0 < \tan\alpha < \infty$;

if $\frac{\pi}{2} < \alpha < \pi$ then $-\infty < \tan\alpha < 0$;

if $\pi < \alpha < \frac{3\pi}{2}$ then $0 < \tan\alpha < \infty$;

if $\frac{3\pi}{2} < \alpha < 2\pi$ then $-\infty < \tan\alpha < 0$.



tan α is an increasing function in every quadrant

Conclusion

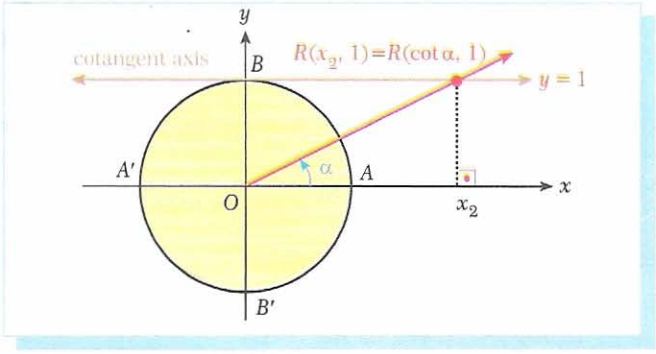
1. For all values of α, the tangent function takes values between $-\infty$ and ∞.
2. The tangent function is not defined for the values $\alpha = \frac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$.
3. In each quadrant, as α increases the tangent function increases.
4. The tangent function is **positive** in the first and third quadrants and **negative** in the second and fourth quadrants.

4. The Cotangent Function

Definition

cotangent of an angle, cotangent function

Let $[OR]$ be the terminal side of an angle α as shown in the figure such that point R lies on the line $y = 1$. Then the abscissa (x-coordinate) of point R on the line $y = 1$ is called the **cotangent of angle α**. It is denoted by **cot α**. The function which matches a real number α to the real number cot α is called the **cotangent function.**

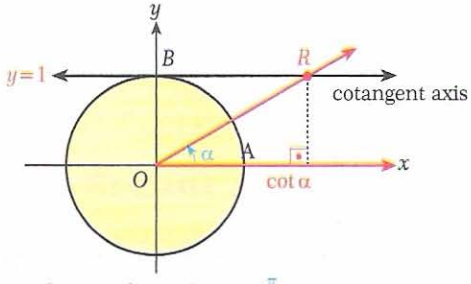


Since the cotangent value of α is the abscissa of the point $R(x, 1)$, the line $y = 1$ can also be called the **cotangent axis.**

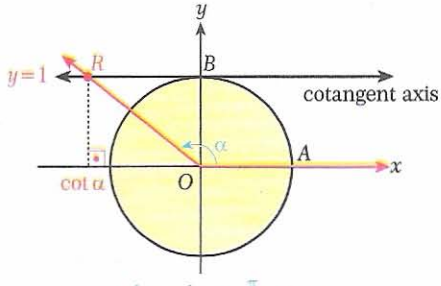
Note

The real numbers $k\pi$ ($k \in \mathbb{Z}$) correspond to the points $A(1, 0)$ and $A'(-1, 0)$ on the unit circle. At these points the terminal side of the angle α coincides with the positive or negative x-axis. Since the x-axis is parallel to the line $y = 1$, the intersection point R does not exist and so at these points the cotangent function is not defined.

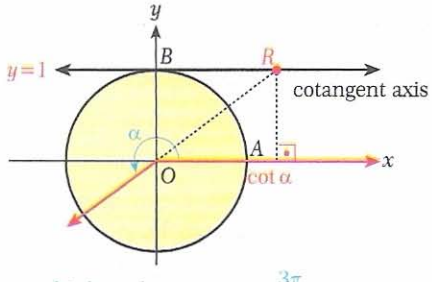
The figures below show how the value of cot α changes as α moves round the circle. Notice that we extend the terminal side of the angle in the third and fourth quadrants to find its intersection with the cotangent line $y = 1$.



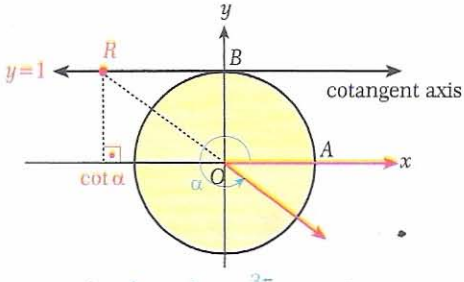
first quadrant: $0 < \alpha < \frac{\pi}{2}$



second quadrant: $\frac{\pi}{2} < \alpha < \pi$



third quadrant: $\pi < \alpha < \frac{3\pi}{2}$



fourth quadrant: $\frac{3\pi}{2} < \alpha < 2\pi$

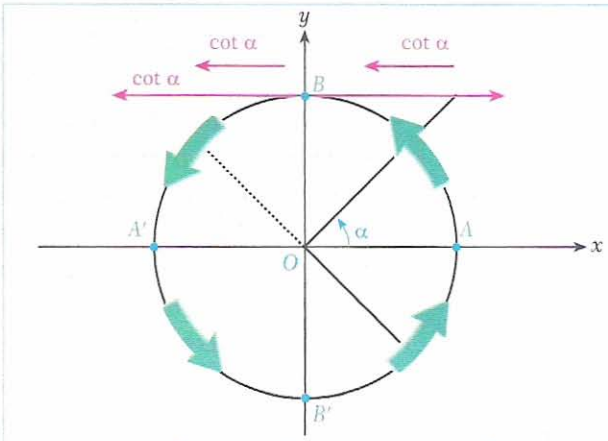
As we can see in the figures:

if $0 < \alpha < \frac{\pi}{2}$ then $0 < \cot \alpha < \infty$;

if $\frac{\pi}{2} < \alpha < \pi$ then $-\infty < \cot \alpha < 0$;

if $\pi < \alpha < \frac{3\pi}{2}$ then $0 < \cot \alpha < \infty$;

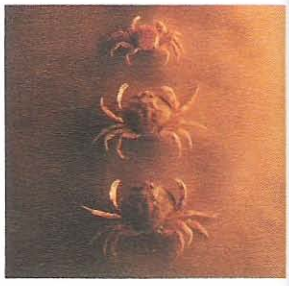
if $\frac{3\pi}{2} < \alpha < 2\pi$ then $-\infty < \cot \alpha < 0$.



cot α is a decreasing function in every quadrant

Conclusion

1. For all values of α, the cotangent function takes values between $-\infty$ and ∞.
2. The cotangent function is not defined for the values $\alpha = k\pi$ $(k \in \mathbb{Z})$.
3. In each quadrant, as α increases the cotangent function **decreases.**
4. The cotangent function is **positive** in the first and third quadrants and **negative** in the second and fourth quadrants.



EXAMPLE 34 Write each set of values in ascending order.

a. $x = \tan 37°$, $y = \tan 36°$, $z = \tan 35°$

b. $a = \cot \frac{5\pi}{9}$, $b = \cot \frac{2\pi}{3}$, $c = \cot \frac{7\pi}{9}$

Solution

a. We know that as α increases, in each quadrant the tangent function is increasing. Therefore $35° < 36° < 37°$ means $\tan 35° < \tan 36° < \tan 37°$. So $z < y < x$.

b. We know that as α increases, in each quadrant the cotangent function is decreasing.

Therefore, $\frac{5\pi}{9} < \frac{2\pi}{3} < \frac{7\pi}{9}$ means $\cot \frac{7\pi}{9} < \cot \frac{2\pi}{3} < \cot \frac{5\pi}{9}$. So $c < b < a$.

EXAMPLE 35 Show that each relation is true by using a unit circle.

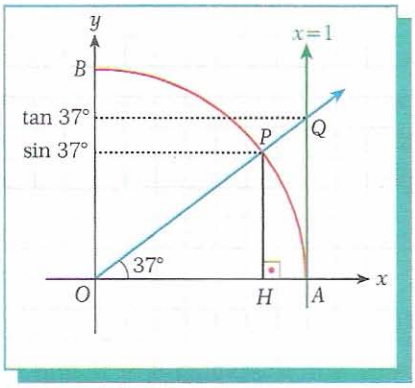
a. $\tan 37° > \sin 37°$

b. $\cos 57° < \cot 57°$

c. $\cos 72° = \sin 18°$

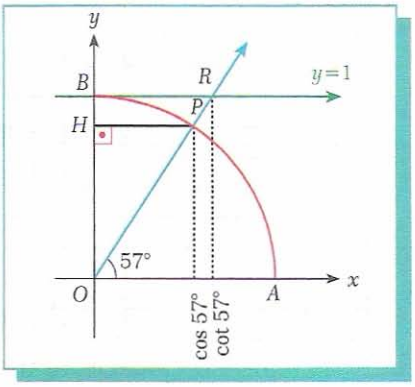
d. $\sin 46° > \cos 46°$

Solution a.



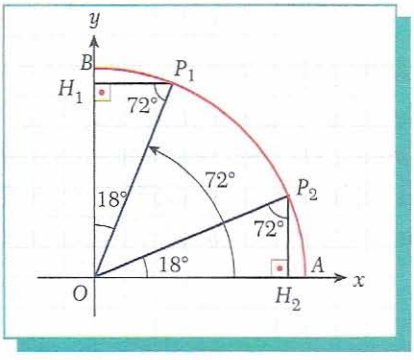
In the figure above,
$AQ > HP$ so $\tan 37° > \sin 37°$.

b.



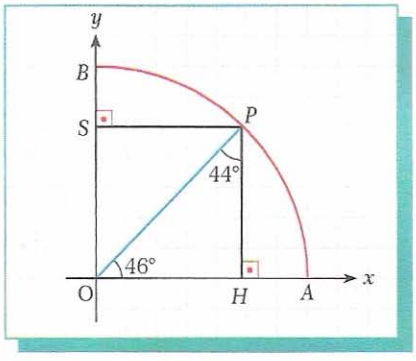
In the figure above,
$HP < BR$ so $\cos 57° < \cot 57°$.

c.



In the figure above,
$P_1H_1 = \cos 72°$, $P_2H_2 = \sin 18°$ and
$\Delta OP_1H_1 \cong \Delta OP_2H_2$.
So $P_1H_1 = P_2H_2$ and
$\cos 72° = \sin 18°$.

d.

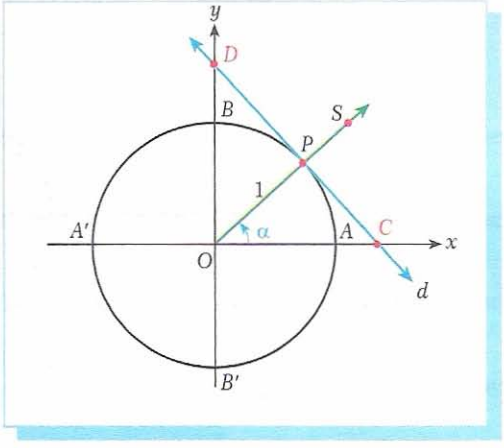


In the figure above, $PH = \sin 46°$ and $PS = \cos 46°$. In a triangle, the larger angle is always opposite the longer side.
So $PH > PS$ and $\sin 46° > \cos 46°$.

5. The Secant and Cosecant Functions

We can also define two more functions by using the unit circle. These are the secant and cosecant functions.

To define the functions, we choose a point P on the unit circle which lies on the terminal side $[OS$ of the angle α, as shown in the figure. Then we draw a tangent line which passes through P and intersects the x-axis and the y-axis at the points C and D respectively. We will define the secant and cosecant functions with the help of points C and D.



Definition

secant of an angle, secant function

Let $[OP]$ be the terminal side of an angle α such that P lies on the unit circle and the tangent at P intersects the x-axis at C. Then the abscissa (x-coordinate) of the point C on the x-axis is called the **secant of angle α**. It is denoted by **sec α**. The function which matches a real number α to the real number sec α is called the **secant function.**

Note

The real numbers $\frac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$ correspond to the points $B(0, 1)$ and $B'(0, -1)$ on the unit circle. At these points, P coincides with the points B or B' and the tangent line at P becomes parallel to the x-axis. Therefore the intersection point C does not exist and so at these points the secant function is not defined.

Definition

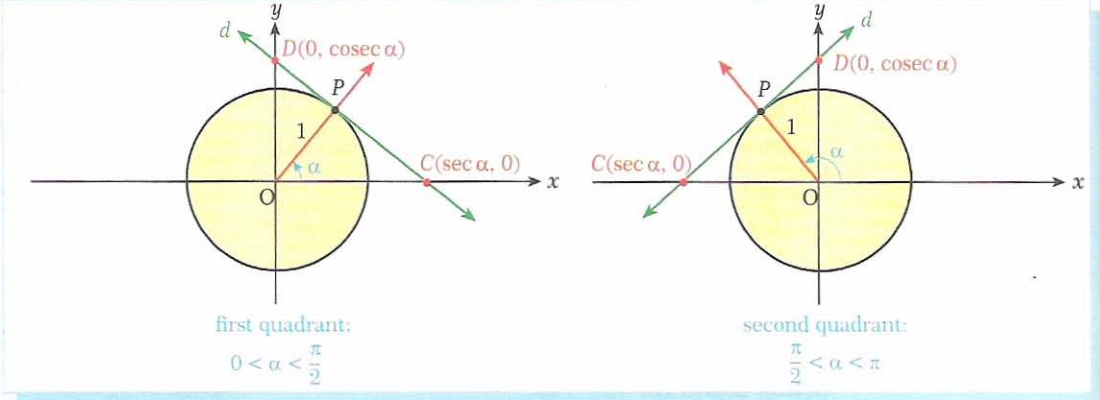
cosecant of an angle, cosecant function

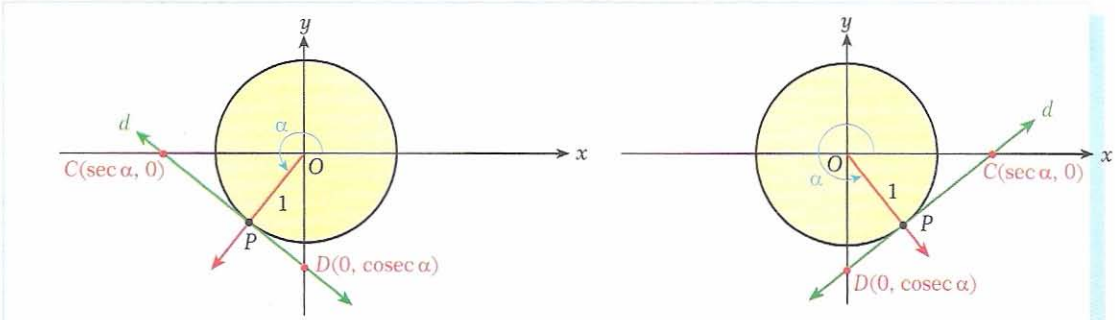
Let $[OP]$ be the terminal side of an angle α such that P lies on the unit circle and the tangent at P intersects the y-axis at D. Then the ordinate (y-coordinate) of point D on the y-axis is called the **cosecant of angle α**. It is denoted by **csc α**. The function which matches a real number α to the real number csc α is called the **cosecant function.**

Note

The real numbers $k\pi$ $(k \in \mathbb{Z})$ correspond to the points $A(1, 0)$ and $A'(-1, 0)$ on the unit circle. At these points, P coincides with the points A or A' and the tangent line at P becomes parallel to the y-axis. Therefore the intersection point D does not exist and so at these points the cosecant function is not defined.

The figures below show how the values of sec α and csc α change as α moves round the circle.





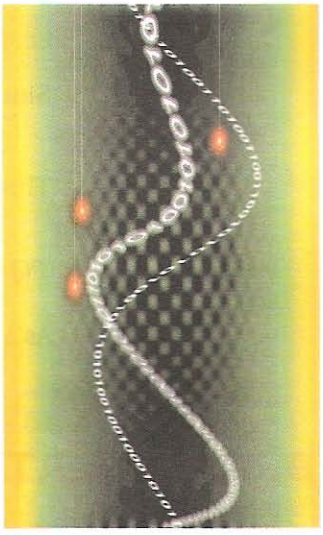
As we can see in the figures:

if $0 < \alpha < \frac{\pi}{2}$ then $1 < \sec \alpha < \infty$ and $1 < \csc \alpha < \infty$;

if $\frac{\pi}{2} < \alpha < \pi$ then $-\infty < \sec \alpha < -1$ and $1 < \csc \alpha < \infty$;

if $\pi < \alpha < \frac{3\pi}{2}$ then $-\infty < \sec \alpha < -1$ and $-\infty < \csc \alpha < -1$;

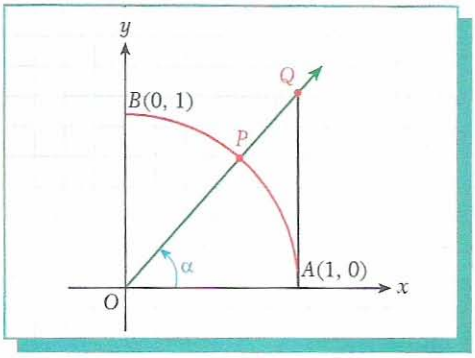
if $\frac{3\pi}{2} < \alpha < 2\pi$ then $1 < \sec \alpha < \infty$ and $-\infty < \csc \alpha < -1$.



Conclusion

1. For all values of α, the secant and cosecant functions take values in $\mathbb{R} - (-1, 1)$.
2. The secant function is not defined for the values $\alpha = \frac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$ and the cosecant function is not defined for the values $\alpha = k\pi$ $(k \in \mathbb{Z})$.
3. In each quadrant, as α increases
 - the secant function **increases** in the first and second quadrants and **decreases** in the third and fourth quadrants.
 - the cosecant function **increases** in the second and third quadrants and **decreases** in the first and fourth quadrants.
4. The secant function is **positive** in the first and fourth quadrants and **negative** in the second and third quadrants.
5. The cosecant function is **positive** in the first and second quadrants and **negative** in the third and fourth quadants.

Example 36 Find the length of PQ in the figure in terms of a trigonometric function of α.



Solution The arc $\overset{\frown}{AB}$ is part of a unit circle.

Let us mark the point H shown in the figure.

Then $OH = \cos\alpha$ and

$OH + HA = 1$, so $\cos\alpha + HA = 1$,

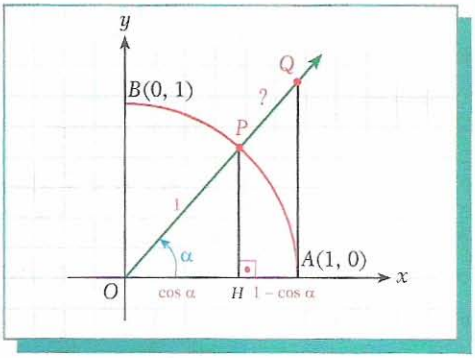
i.e. $HA = 1 - \cos\alpha$.

Since $PH \parallel QA$, $\dfrac{OP}{PQ} = \dfrac{OH}{HA}$.

This gives $\dfrac{1}{PQ} = \dfrac{\cos\alpha}{1-\cos\alpha}$, i.e. $PQ = \dfrac{1-\cos\alpha}{\cos\alpha}$,

$PQ = \dfrac{1}{\cos\alpha} - 1$.

As a result, $PQ = \sec\alpha - 1$.



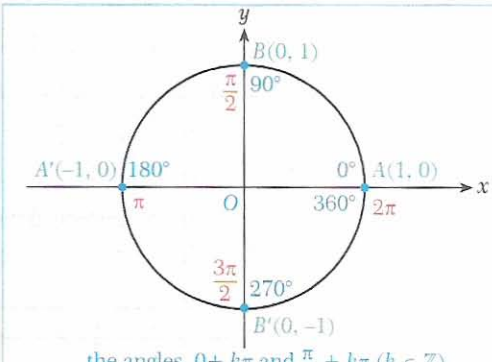
B. CALCULATING TRIGONOMETRIC VALUES

1. Trigonometric Values of Quadrantal Angles

Definition

quadrantal angle

If the terminal side of an angle coincides with a coordinate axis then the angle is called a **quarantal angle**. If an angle is not quadrantal, it is called a **nonquadrantal** angle.



the angles $0 + k\pi$ and $\frac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$ are quadrantal angles

We can calculate the trigonometric values of quadrantal angles by observing at the points at which the terminal sides of the angles intersect the unit circle.

Trigonometric Values of Quadrantal Angles

θ in degrees	θ in radians	$\sin\theta$ (y)	$\cos\theta$ (x)	$\tan\theta$ (y/x)	$\cot\theta$ (x/y)	$\sec\theta$ $(1/x)$	$\csc\theta$ $(1/y)$
0°	0	0	1	0	undefined	1	undefined
90°	$\frac{\pi}{2}$	1	0	undefined	0	undefined	1
180°	π	0	–1	0	undefined	–1	undefined
270°	$\frac{3\pi}{2}$	–1	0	undefined	0	undefined	–1
360°	2π	0	1	0	undefined	1	undefined

EXAMPLE 37 Evaluate each expression.

a. $\sin 0° + \cos 270° + \tan 180° - \cot 90°$

b. $\sin 90° - \tan 180° + \cot 270° - \cos 180°$

Solution We can use the table above.

a. $0 + 0 + 0 - 0 = 0$

b. $1 - 0 + 0 - (-1) = 2$

EXAMPLE 38 Evaluate each expression.

a. $\sin\pi + \cos\frac{\pi}{2} + \tan 0 - \cot\frac{3\pi}{2}$ b. $\sin\frac{3\pi}{2} - \tan\pi - \cot\frac{\pi}{2} - \cos 0$

Solution a. $0 + 0 + 0 - 0 = 0$

b. $-1 - 0 - 0 - 1 = -2$

EXAMPLE 39 Evaluate $\sin 1710° - \cos 2520° + \cot 450° - \tan 900°$.

Solution The angles are all greater than 360° so we begin by finding the primary directed angle of each term.

$\sin 1710° = \sin(4 \cdot 360° + 270°) = \sin 270°$

$\cos 2520° = \cos(7 \cdot 360° + 0°) = \cos 0°$

$\cot 450° = \cot(360° + 90°) = \cot 90°$

$\tan 900° = \tan(2 \cdot 360° + 180°) = \tan 180°$

Hence,

$$\sin 1710° - \cos 2520° + \cot 450° - \tan 900° = \sin 270° - \cos 0° + \cot 90° - \tan 180°$$
$$= -1 - 1 + 0 - 0 = -2.$$

Check Yourself 11

Evaluate each expression by using the table of trigonometric values for quadrantal angles.

1. $2 \cdot \sin 180° + \tan \pi + 5 \cdot \cot 270° + 3 \cdot \cos 180°$.

2. $$\frac{4 \cdot \cos 0 - 10 \cdot \sin \frac{\pi}{2}}{2 \cdot \sin 270° - \cos \pi}$$

Answers

1. –3 2. 6

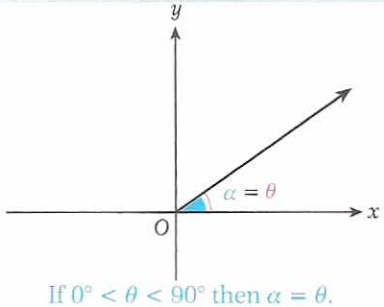
2. Using a Reference Angle

In this section we will learn how to find the trigonometric ratios of any angle in terms of the trigonometric ratios of a corresponding acute angle. We will use the special trigonometric ratios for 30°, 45° and 60° angles which we studied in section 1.2.

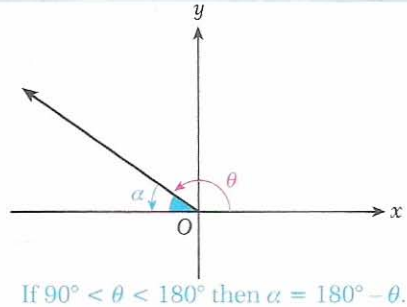
Definition reference angle

The positive acute angle α which is formed by the terminal side of a nonquadrantal angle θ and the x-axis is called the **reference angle** for θ.

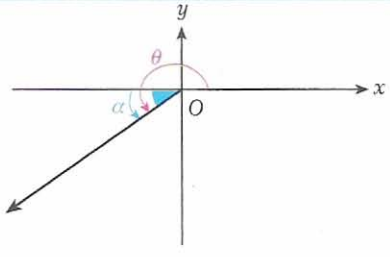
Look at the figures. They show how to find the reference angle α for an angle θ in each quadrant.



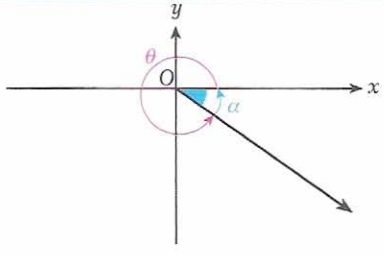
If $0° < \theta < 90°$ then $\alpha = \theta$.



If $90° < \theta < 180°$ then $\alpha = 180° - \theta$.



If $180° < \theta < 270°$ then $\alpha = \theta - 180°$.



If $270° < \theta < 360°$ then $\alpha = 360° - \theta$.

EXAMPLE 40 Find the reference angle for each angle.

a. 30° b. 150° c. 215° d. 317°

Solution a. Since $0 < 30° < 90°$, the reference angle for 30° is 30°.

b. Since $90° < 150° < 180°$, the reference angle for 150° is $180° - 150° = 30°$.

c. Since $180° < 215° < 270°$, the reference angle for 270° is $270° - 215° = 55°$.

d. Since $270° < 137° < 360°$, the reference angle for 317° is $360° - 317° = 43°$.



Now that we can calculate reference angles we are ready to calculate the trigonometric value of a nonquadrantal angle. To do this, follow the steps:

1. Find the primary directed angle of the nonquadrantal angle and determine its quadrant.
2. Determine the sign of the function in this quadrant.
3. Calculate the reference angle for the given angle.
4. Find the trigonometric value of the reference angle and use it with the sign from step 2.

EXAMPLE 41 Find each trigonometric value by using a reference angle.

a. cos 135° b. sin 330° c. sec 240° d. csc 120°

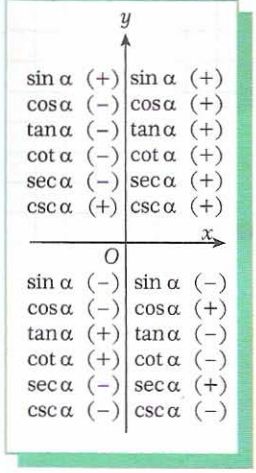
e. sin 510° f. cos 945° g. tan (–930°) h. cot (–675°)

Solution a. 1. 135° is already a primary directed angle and it is in the second quadrant.

2. In the second quadrant, the cosine function is negative.

3. The reference angle is $\alpha = 180° - 135° = 45°$.

4. $\cos 135° = -\cos 45° = -\dfrac{1}{\sqrt{2}}$



b. 1. 330° is already a primary directed angle and it is in the fourth quadrant.

2. In the fourth quadrant, the sine function is negative.

3. The reference angle is $\alpha = 360° - 330° = 30°$.

4. $\sin 330° = -\sin 30° = -\dfrac{1}{2}$

c. 1. 240° is a primary directed angle, third quadrant

2. In the third quadrant, the secant function is negative.

3. The reference angle is $\alpha = 240° - 180° = 60°$.

4. $\sec 240° = -\sec 60° = -2$

d. 1. 120°: primary directed angle, second quadrant

2. In the second quadrant, the cosecant function is positive.

3. $\alpha = 180° - 120° = 60°$

4. $\csc 120° = \csc 60° = \frac{2}{\sqrt{3}}$

e. 1. $510° = 360° + 150°$. So the primary directed angle of 510° is 150° and it is in the second quadrant.

2. In the second quadrant, the sine function is positive.

3. $\alpha = 180° - 150° = 30°$

4. $\sin 510° = \sin 150° = \sin 30° = \frac{1}{2}$

f. 1. $945° = (2 \cdot 360°) + 225°$. So the primary directed angle is 225° and it is in the third quadrant.

2. In the third quadrant, the cosine function is negative.

3. $\alpha = 225° - 180° = 45°$

4. $\cos 945° = \cos 225° = -\cos 45° = -\frac{1}{\sqrt{2}}$

g. 1. $-930° = (-3 \cdot 360°) + 150°$. So the primary directed angle is 150° and it is in the second quadrant.

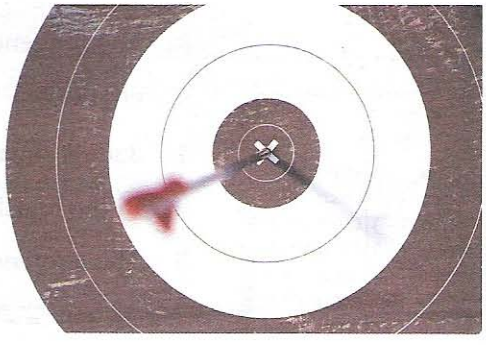
2. In the second quadrant, the tangent function is negative.

3. $\alpha = 180° - 150° = 30°$

4. $\tan(-930°) = \tan 150°$

$= -\tan 30°$

$= -\frac{1}{\sqrt{3}}$



h. 1. $-675° = (-2 \cdot 360°) + 45°$.

So the primary directed angle is 45° and it is in the first quadrant.

2. In the first quadrant, the cotangent function is positive.

3. $\alpha = 45° - 0° = 45°$

4. $\cot(-675°) = \cot 45° = 1$

EXAMPLE 42 Find the each trigonometric value by using a reference angle.

a. $\cot\frac{7\pi}{6}$ b. $\tan\frac{31\pi}{4}$ c. $\sin\left(-\frac{25\pi}{3}\right)$ d. $\cos\left(-\frac{47\pi}{4}\right)$

Solution a. 1. $\frac{7}{6}<2$ so $\frac{7\pi}{6}<2\pi$. So this is a primary directed angle. Moreover, $\frac{7\pi}{6}=\pi+\frac{\pi}{6}$ so the angle is in the third quadrant.

2. In the third quadrant, the cotangent function is positive.

3. $\alpha=\frac{7\pi}{6}-\pi=\frac{\pi}{6}$

4. $\cot\frac{7\pi}{6}=\cot\frac{\pi}{6}=\sqrt{3}$

b. 1. Since $\frac{31}{4}>2$, we need to write the angle as a primary directed angle:

$\frac{31\pi}{4}=(3\cdot 2\pi)+\frac{7\pi}{4}$. So the primary directed angle is $\frac{7\pi}{4}$ and it is in the fourth quadrant.

2. In the fourth quadrant, the tangent function is negative.

3. $\alpha=2\pi-\frac{7\pi}{4}=\frac{\pi}{4}$

4. $\tan\frac{31\pi}{4}=\tan\frac{7\pi}{4}=-\tan\frac{\pi}{4}=-1$

c. 1. Since $-\frac{25}{3}<-2$, we need to write the angle as a primary directed angle.

$-\frac{25\pi}{3}=(-5\cdot 2\pi)+\frac{5\pi}{3}$. So the primary directed angle is $\frac{5\pi}{3}$ and it is in the fourth quadrant.

2. In the fourth quadrant, the sine function is negative.

3. $\alpha=2\pi-\frac{5\pi}{3}=\frac{\pi}{3}$

4. $\sin\left(-\frac{25\pi}{3}\right)=\sin\frac{5\pi}{3}=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$

d. 1. $-\frac{47}{4}<-2$ so we need to find the primary directed angle: $-\frac{47\pi}{4}=-6\cdot(2\pi)+\frac{\pi}{4}$.
The angle is in the first quadrant.

2. In the first quadrant, the cosine function is positive.

3. $\alpha=\frac{\pi}{4}-0=\frac{\pi}{4}$

4. $\cos\left(-\frac{47\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

Check Yourself 12

1. Find the reference angle for each angle.

 a. 890° b. 5000° c. –850° d. –2500°

 e. $\frac{32\pi}{7}$ f. $\frac{103\pi}{6}$ g. $-\frac{50\pi}{11}$ h. $-\frac{11\pi}{5}$

2. Find each trigonometric value by using a reference angle.

 a. sin 570° b. tan 405° c. cos (–2550°) d. cot (–7950°)

 e. $\cot\frac{19\pi}{3}$ f. $\cos\frac{27\pi}{4}$ g. $\sin\left(-\frac{45\pi}{4}\right)$ h. $\tan\left(-\frac{25\pi}{6}\right)$

Answers

1. a. 10° b. 40° c. 50° d. 20° e. $\frac{3\pi}{7}$ f. $\frac{\pi}{6}$ g. $\frac{5\pi}{11}$ h. $\frac{\pi}{5}$

2. a. $-\frac{1}{2}$ b. 1 c. $\frac{\sqrt{3}}{2}$ d. $-\sqrt{3}$ e. $\frac{\sqrt{3}}{3}$ f. $-\frac{\sqrt{2}}{2}$ g. $\frac{\sqrt{2}}{2}$ h. $-\frac{\sqrt{3}}{3}$

3. Calculating Ratios from a Given Ratio

In this section we will learn how to find all the trigonometric ratios of an angle from a single given ratio. In solving such problems we will use our knowledge of trigonometric ratios in right triangles, the sign of a trigonometric function and the fundamental trigonometric identities. Let us look at an example.

EXAMPLE 43 For each trigonometric ratio in the given quadrant, find the five other trigonometric ratios for same angle.

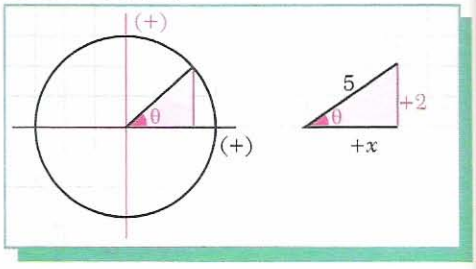
a. $\sin\theta = \frac{2}{5}$, $\theta \in (0°, 90°)$ b. $\cos\theta = -\frac{1}{3}$, $\theta \in (90°, 180°)$

c. $\tan\theta = \frac{7}{4}$, $\theta \in \left(\pi, \frac{3\pi}{2}\right)$ d. $\cot\theta = -6$, $\theta \in \left(\frac{3\pi}{2}, 2\pi\right)$

Solution We will use the abbreviations *opp*, *adj* and *hyp* to mean the opposite side, adjacent side and hypotenuse of a triangle with respect to the angle θ.

a. The angle is in the first quadrant. In this quadrant, both axes are positive and so the sine and cosine values will be positive. By the Pythagorean Theorem,

$$hyp^2 = adj^2 + 2^2$$
$$5^2 = adj^2 + 2^2$$
$$adj^2 = 25 - 4$$
$$adj = \pm\sqrt{21}.$$



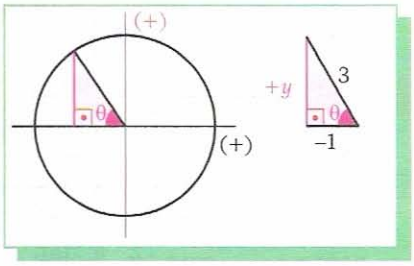
We choose the positive value for the first quadrant:

$adj = \sqrt{21}$.

As a result, $\cos\theta = \frac{\sqrt{21}}{5}$, $\tan\theta = \frac{2}{\sqrt{21}}$, $\cot\theta = \frac{\sqrt{21}}{2}$, $\sec\theta = \frac{5}{\sqrt{21}}$, $\csc\theta = \frac{5}{2}$.

b. The angle is in the second quadrant. In this quadrant the x-axis is negative and the y-axis is positive. The cosine function is related to the x-axis, and so

$\cos\theta = -\frac{1}{3}$ can be taken as $\cos\theta = \frac{adj}{hyp} = \frac{-1}{+3}$.

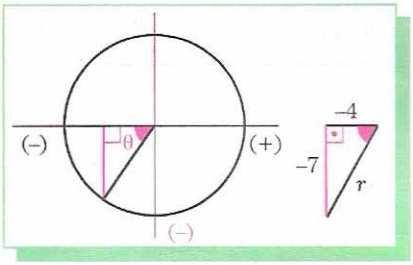


By the Pythagorean Theorem, $opp = \sqrt{8} = 2\sqrt{2}$.

As a result,

$$\sin\theta = \frac{2\sqrt{2}}{3},\ \tan\theta = \frac{2\sqrt{2}}{-1} = -2\sqrt{2},\ \cot\theta = \frac{-1}{2\sqrt{2}},\ \sec\theta = \frac{3}{-1} = -3,\ \csc\theta = \frac{3}{2\sqrt{2}}.$$

c. Similarly, $\tan\theta = \frac{opp}{adj} = \frac{7}{4} = \frac{-7}{-4}$ using the signs of the axes in the third quadrant. By the Pythagorean Theorem, $hyp = \sqrt{65}$. As a result,



$$\sin\theta = \frac{-7}{\sqrt{65}},\ \cos\theta = \frac{-4}{\sqrt{65}},\ \cot\theta = \frac{-4}{-7} = \frac{4}{7},$$

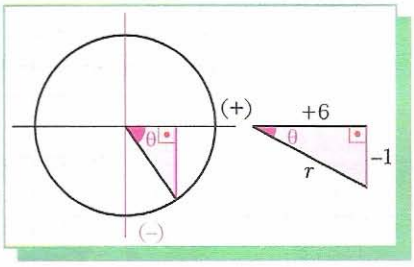
$$\sec\theta = \frac{\sqrt{65}}{-4},\ \csc\theta = \frac{\sqrt{65}}{-7}.$$

d. Finally, $\cot\theta = \frac{adj}{opp} = -6 = \frac{6}{-1}$ using the signs of the axes in the fourth quadrant.

By the Pythagorean Theorem, $hyp = \sqrt{37}$.

As a result, $\sin\theta = \frac{-1}{\sqrt{37}},\ \cos\theta = \frac{6}{\sqrt{37}},$

$$\tan\theta = \frac{-1}{6},\ \sec\theta = \frac{\sqrt{37}}{6},\ \csc\theta = \frac{\sqrt{37}}{-1} = -\sqrt{37}.$$





Check Yourself 13

1. Find the sine, cosine and tangent ratios of each angle without using a trigonometric table or calculator.

 a. $\theta = 315°$ b. $\theta = \frac{2\pi}{3}$ c. $\theta = -900°$ d. $\theta = \frac{63\pi}{2}$

2. a. $\tan\theta = 5$, $\theta \in (180°, 270°)$ are given.
 Find $\sin\theta$ and $\cos\theta$.

 b. $\sec\theta = -10$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$ are given.
 Find $\sin\theta$ and $\tan\theta$.

Answers

1. a. $\sin 315° = -\frac{1}{\sqrt{2}}$, $\cos 315° = \frac{1}{\sqrt{2}}$, $\tan 315° = -1$

 b. $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\tan\frac{2\pi}{3} = -\sqrt{3}$

 c. $\sin 900° = 0$, $\cos 900° = -1$, $\tan 900° = 0$

 d. $\sin\frac{63\pi}{2} = -1$, $\cos\frac{63\pi}{2} = 0$, $\tan\frac{63\pi}{2}$ is undefined

2. a. $\sin\theta = -\frac{5}{\sqrt{26}}$ $\cos\theta = -\frac{1}{\sqrt{26}}$

 b. $\sin\theta = \frac{3\sqrt{11}}{10}$ $\tan\theta = -3\sqrt{11}$

4. Trigonometric Values of Other Angles

Our trigonometric calculations up to now have mostly used our knowledge of the trigonometric values of the special angles 0°, 30°, 45°, 90°, 270°, 360° etc. We have calculated and verified these values using a unit circle and a right triangle. However, the trigonometric values of other angles such as 11°, 27°, 105° etc. cannot be calculated in this way.

To find these values we can use either a trigonometric table or (more commonly) a calculator. Let us look at each method in turn.

a. Using a trigonometric table

A trigonometric table is a list of trigonometric values for a range of angle measures. The table on the next page shows the sine, cosine and tangent values for angles between 0° and 90°, in unit increments. The values are approximated to four decimal places.

Trigonometric Table

α angle	sin	cos	tan
0	0.0000	1.0000	0.0000
1	0.0175	0.9998	0.0175
2	0.0349	0.9994	0.0349
3	0.0523	0.9986	0.0524
4	0.0698	0.9976	0.0699
5	0.0872	0.9962	0.0875
6	0.1045	0.9945	0.1051
7	0.1219	0.9925	0.1228
8	0.1392	0.9903	0.1405
9	0.1564	0.9877	0.1584
10	0.1736	0.9848	0.1763
11	0.1908	0.9816	0.1944
12	0.2079	0.9781	0.2126
13	0.2250	0.9744	0.2309
14	0.2419	0.9703	0.2493
15	0.2588	0.9659	0.2679
16	0.2756	0.9613	0.2867
17	0.2924	0.9563	0.3057
18	0.3090	0.9511	0.3249
19	0.3256	0.9455	0.3443
20	0.3420	0.9397	0.3640
21	0.3584	0.9336	0.3839
22	0.3746	0.9272	0.4040
23	0.3907	0.9205	0.4245
24	0.4067	0.9135	0.4452
25	0.4226	0.9063	0.4663
26	0.4384	0.8988	0.4877
27	0.4540	0.8910	0.5095
28	0.4695	0.8829	0.5317
29	0.4848	0.8746	0.5543
30	0.5000	0.8660	0.5774
31	0.5150	0.8572	0.6009
32	0.5299	0.8480	0.6249
33	0.5446	0.8387	0.6494
34	0.5592	0.8290	0.6745
35	0.5736	0.8192	0.7002
36	0.5878	0.8090	0.7265
37	0.6018	0.7986	0.7536
38	0.6157	0.7880	0.7813
39	0.6293	0.7771	0.8098
40	0.6428	0.7660	0.8391
41	0.6561	0.7547	0.8693
42	0.6691	0.7431	0.9004
43	0.6820	0.7314	0.9325
44	0.6947	0.7193	0.9657
45	0.7071	0.7071	1.0000

α angle	sin	cos	tan
45	0.7071	0.7071	1.0000
46	0.7193	0.6947	1.0355
47	0.7314	0.6820	1.0724
48	0.7431	0.6691	1.1106
49	0.7547	0.6561	1.1504
50	0.7660	0.6428	1.1918
51	0.7771	0.6293	1.2349
52	0.7880	0.6157	1.2799
53	0.7986	0.6018	1.3270
54	0.8090	0.5878	1.3764
55	0.8192	0.5736	1.4281
56	0.8290	0.5592	1.4826
57	0.8387	0.5446	1.5399
58	0.8480	0.5299	1.6003
59	0.8572	0.5150	1.6643
60	0.8660	0.5000	1.7321
61	0.8746	0.4848	1.8040
62	0.8829	0.4695	1.8807
63	0.8910	0.4540	1.9626
64	0.8988	0.4384	2.0503
65	0.9063	0.4226	2.1445
66	0.9135	0.4067	2.2460
67	0.9205	0.3907	2.3559
68	0.9272	0.3746	2.4751
69	0.9336	0.3584	2.6051
70	0.9397	0.3420	2.7475
71	0.9455	0.3256	2.9042
72	0.9511	0.3090	3.0777
73	0.9563	0.2924	3.2709
74	0.9613	0.2756	3.4874
75	0.9659	0.2588	3.7321
76	0.9703	0.2419	4.0108
77	0.9744	0.2250	4.3315
78	0.9781	0.2079	4.7046
79	0.9816	0.1908	5.1446
80	0.9848	0.1736	5.6713
81	0.9877	0.1564	6.3138
82	0.9903	0.1392	7.1154
83	0.9925	0.1219	8.1443
84	0.9945	0.1045	9.5144
85	0.9962	0.0872	11.4301
86	0.9976	0.0698	14.3007
87	0.9986	0.0523	19.0811
88	0.9994	0.0349	28.6363
89	0.9998	0.0175	57.2900
90	1.0000	0.0000	undefined

EXAMPLE 44 Find the values of cos 11° and tan 63° using a trigonometric table.

Solution In the table, we look down the angle column to find 11° and then move across to find the value in the cosine column. We can see that cos 11° ≅ 0.9816.
Similarly, we find 63° in the right-hand half of the table and look across to the value in the tangent column. The answer is tan 63° ≅ 1.9626.
Notice that we use the ≅ sign to show that these values are approximate, as the values in the table are only given to four decimal places.

EXAMPLE 45 Find the approximate value of sin 73.25° using a trigonometric table.

Solution The trigonometric table only gives the trigonometric values of whole numbers. We can find the approximate trigonometric value of a decimal angle by assuming that the sine function increases linearly and using direct proportion.

From the table,

$\sin 73° \cong 0.9563$

$\sin 74° \cong 0.9613.$

For 1° = 60′ the sine value increases by 0.9613 − 0.9563 = 0.0050.

For 25′ the proportional increase is therefore

$$\begin{array}{ll} 60' & 0.0050 \\ 25' & x \end{array}$$

$$x = \frac{25 \cdot 0.0050}{60} \cong 0.0021.$$

We add this value to the sine of 73° to get sin 73.25° ≅ 0.9563 + 0.0021 = 0.9584.

Of course, the sine function does not actually increase linearly in this way. However, its change over one degree is approximately linear, and the question only asks for an approximate value.

EXAMPLE 46 cot α = 1.13 is given. Use a trigonometric table to find the approximate value of the angle α in degree-minute form.

Solution We cannot find the cotangent value 1.13 directly in the table.

The value is between the values cot 41° = tan 49° = 1.1504 and cot 42° = tan 48° = 1.1106.

Since 1.1106 < 1.13 < 1.1504, 41° < α < 42°.

1. The cotangent function is decreasing in every quadrant, so cot 41° > cot 42°.
2. cot 41° = tan 49°
 cot 42° = tan 48°

For $1° = 60'$ the cotangent value decreases by $1.1504 - 1.1106 = 0.0398$. The decrease in value between $41°$ and α is $1.1504 - 1.13 = 0.0204$.

To get the approximate value, we assume that the cotangent function is linear in the interval $[41°, 42°]$.

$$\begin{array}{cc} 60' & 0.0398 \\ x & 0.0204 \end{array}$$

$$x \cong \frac{0.0204 \cdot 60}{0.0398} \cong 31'$$

We add this value to the cotangent of $41°$ to get $\cot 41°31' \cong 1.13$. As a result, $\alpha \cong 41°31'$.

b. Using a calculator

The easiest way to find a trigonometric ratio today is with the help of a secientific calculator.

Note

The steps shown in these examples may be slightly different on different models of calculator. The examples show results rounded to four decimal places, although your calculator will round to more than this.

EXAMPLE 47 Find $\cos 53°$ on a calculator.

Solution First of all, check that the calculator is set for degree input.

Step	Keys pressed	Display
Enter 53	5 3	53
Find the cosine	COS	0.6018...

EXAMPLE 48 Find $\sin 28.25°$ on a calculator.

Solution First of all, check that the calculator is set for degree input.

Step	Keys pressed	Display
Enter 28.25	2 8 . 2 5	28.25
Find the sine	SIN	0.4733...

EXAMPLE 49 Find the approximate value of the angle α in degrees if $\tan \alpha = 10.07$.

Solution

Step	Keys pressed	Display
Enter 10.07	1 0 . 0 7	10.07
Find α	INV TAN	84.3288...

The abbreviation **inv** or **arc** means the inverse of the trigonometric function. We want to find the value of α such that $\tan \alpha = 10.07$, so we need to use the inverse tangent function.

Check Yourself 14

1. Find each trigonometric value rounded to four significant figures.

 a. $\sin 36°$ b. $\cos 44.16°$

2. Find the angle α for each trigonometric value, rounded to the nearest degree.

 a. $\tan \alpha = 0.3057°$ b. $\sec \alpha = 1.5243°$

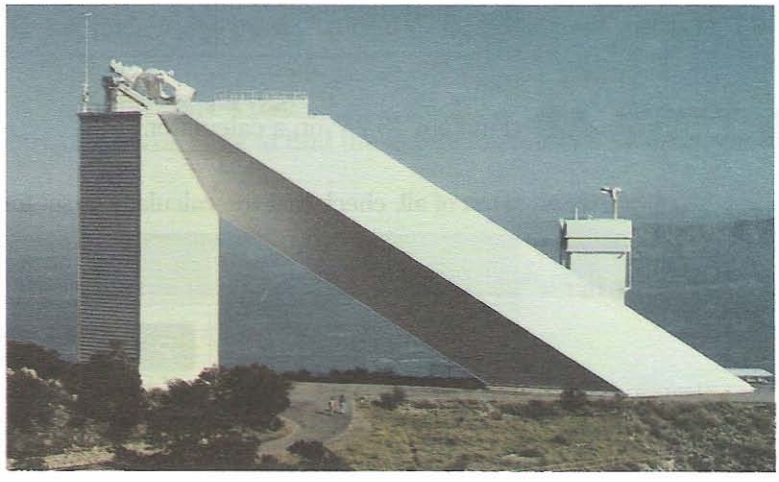
3. Find the angle α for each trigonometric value, in degrees rounded to two significant figures.

 a. $\csc \alpha = 5$ b. $\cot \alpha = 1.2345$

Significant figures are the minimum number of digits needed to write a given value in scientific notation without loss of accuracy.

Answers

1. a. 0.5878
 b. 0.7174
2. a. 17°
 b. 49°
3. a. 11°
 b. 39°



EXERCISES 3

A. Trigonometric Functions

1. Complete the table of trigonometric functions on the unit circle.

Trigonometric functions on the unit circle	
Function	Corresponding real number
$\cos\theta$	x
	y
$\tan\theta$	
	x/y
	$1/x$
$\csc\theta$	

2. Show that the following points lie on the unit circle.

 a. $A\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ b. $B\left(-\frac{2}{7}, -\frac{3\sqrt{5}}{7}\right)$

3. Determine whether or not each point lies on the unit circle.

 a. $C\left(\frac{1}{2}, \frac{3}{4}\right)$ b. $D\left(\frac{1}{2}, \frac{1}{2}\right)$

4. Order each set of trigonometric values.

 a. $x = \sin 17°$, $y = \sin 57°$, $z = \sin 73°$

 b. $x = \cos 35°$, $y = \cos 55°$, $z = \sin 63°$, $p = \cos 89°$

 c. $x = \tan 98°$, $y = \tan 101°$, $z = \tan 123°$, $p = \cos 172°$

B. Calculating Trigonometric Values

5. Complete the table.

Trigonometric Values of Quadrantal Angles							
θ in Degree	θ in Radian	sin θ (y)	cos θ (x)	tan θ (y/x)	cot θ (x/y)	sec θ (1/x)	csc θ (1/y)
0°			1				
	$\frac{\pi}{2}$	1		undefined	0	undefined	
180°		0	−1		undefined		undefined
270°			0	undefined	0	undefined	−1
	2π	0				1	undefined

6. Evaluate each expression without using a trigonometric table or calculator.

 a. $\sin 180° + \cos 270° + \tan 360° + \cot 90°$

 b. $\sin 90° - \cos 270° - (\tan 180° \cdot \cot 270°)$

7. Find the reference angle for each angle.

 a. 12° b. 112° c. 212° d. 312°

 e. 50° f. 150° g. 250° h. 350°

8. Find the reference angle for each angle.

 a. −25° b. −140° c. −245° d. −305°

 e. −5° f. −95° g. −260° h. −320°

9. Find the reference angle for each angle.

 a. 1000° b. 3456° c. −3000° d. −7890°

 e. 2000° f. 6789° g. −1000° h. −2345°

10. Find the reference angle for each angle.

 a. $\frac{\pi}{11}$ b. $\frac{7\pi}{12}$ c. $\frac{18\pi}{13}$ d. $\frac{25\pi}{14}$

 e. $\frac{2\pi}{13}$ f. $\frac{9\pi}{15}$ g. $\frac{22\pi}{17}$ h. $\frac{36\pi}{19}$

11. Find the reference angle for each angle.

 a. $-\frac{\pi}{8}$ b. $-\frac{7\pi}{10}$ c. $-\frac{13\pi}{12}$ d. $-\frac{23\pi}{14}$

 e. $-\frac{\pi}{6}$ f. $-\frac{8\pi}{9}$ g. $-\frac{17\pi}{12}$ h. $-\frac{28\pi}{15}$

12. Find the reference angle for each angle.

a. $\frac{73\pi}{6}$ b. $\frac{2019\pi}{9}$ c. $-\frac{101\pi}{13}$ d. $-\frac{1001\pi}{15}$

e. $\frac{57\pi}{7}$ f. $\frac{1007\pi}{73}$ g. $-\frac{602\pi}{98}$ h. $-\frac{1009\pi}{99}$

13. Complete the table with + and – signs.

Signs of the Trigonometric Values of Nonquadrantal Angles

Quadrant, axis		$\sin\theta=\frac{y}{r}$	$\cos\theta=\frac{x}{r}$	$\tan\theta=\frac{y}{x}$	$\cot\theta=\frac{x}{y}$	$\sec\theta=\frac{r}{x}$	$\csc\theta=\frac{r}{y}$
I	$x>0$ $y>0$	+			+		
II	$x<0$ $y>0$		–	–			+
III	$x<0$ $y<0$		–	+	+	–	
IV	$x>0$ $y<0$	–			–	+	–

14. Find each trigonometric value by using a reference angle.

a. sin 120° b. cos 240°

c. tan 315° d. cot 135°

e. sin 855° f. cos 3660°

g. tan 2025° h. cot 1410°

15. Find each trigonometric value by using a reference angle.

a. sin (–225°) b. cos (–150°)

c. tan (–300°) d. cot (–30°)

e. sin (–1590°) f. cos (–675°)

g. tan (–9045°) h. cot (–600°)

16. Find each trigonometric value by using a reference angle.

a. $\sin\frac{5\pi}{6}$ b. $\cos\frac{\pi}{3}$

c. $\tan\frac{5\pi}{4}$ d. $\cot\frac{11\pi}{6}$

e. $\sin\left(-\frac{\pi}{4}\right)$ f. $\cos\left(-\frac{2\pi}{3}\right)$

g. $\tan\left(-\frac{5\pi}{4}\right)$ h. $\cot\left(-\frac{7\pi}{6}\right)$

17. Find each trigonometric value by using a reference angle.

a. $\sin\frac{67\pi}{6}$ b. $\cos\frac{100\pi}{3}$

c. $\tan\frac{55\pi}{4}$ d. $\cot\frac{607\pi}{6}$

e. $\sin\left(-\frac{83\pi}{4}\right)$ f. $\cos\left(-\frac{202\pi}{3}\right)$

g. $\tan\left(-\frac{151\pi}{4}\right)$ h. $\cot\left(-\frac{89\pi}{6}\right)$

18. Evaluate the expressions.

a. $\cos 45° + \cos 330° + \cos 150° + \cos 315°$

b. $\cot\frac{5\pi}{6} - \sin\frac{5\pi}{4} + \tan\frac{\pi}{3} - \cos\frac{7\pi}{4}$

c. $\sec 300° + \tan 585° + \cot 765° + \csc 1230°$

19. Evaluate each expression given that α is an acute angle.

a. $\sin(180° + \alpha) - \cos(270° + \alpha) + \tan(360° + \alpha) + \cot(900° + \alpha)$

b. $\sin(\pi-\alpha)+\sin(\pi+\alpha)+\cot(\frac{\pi}{2}-\alpha)+\cot(\frac{\pi}{2}+\alpha)$

20. For each trigonometric ratio in the given quadrant, find the other trigonometric ratios for θ.

a. $\sin\theta=\frac{\sqrt{3}}{2}$, $\theta\in(0°, 90°)$

b. $\cos\theta=-\frac{\sqrt{2}}{2}$, $\theta\in(90°, 180°)$

c. $\tan\theta=\frac{5}{4}$, $\theta\in(180°, 270°)$

d. $\cot\theta=-\frac{2}{3}$, $\theta\in(270°, 360°)$

21. For each trigonometric ratio in the given quadrant, find the other trigonometric ratios for θ.

a. $\sin\theta = -\frac{\sqrt{2}}{2}$, $\theta \in \left(\frac{3\pi}{2}, 2\pi\right)$

b. $\cos\theta = -\frac{24}{25}$, $\theta \in \left(\pi, \frac{3\pi}{2}\right)$

c. $\tan\theta = -4$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$

d. $\cot\theta = 7$, $\theta \in \left(0, \frac{\pi}{2}\right)$

22. α is an acute angle.

a. If $\frac{3\cdot\sin\alpha + 1}{4 - 5\cdot\sin\alpha} = \frac{2}{5}$, what is $\sin\alpha$?

b. If $\frac{\cos\alpha - 2}{7\cdot\cos\alpha + 11} = -\frac{1}{6}$, what is $\cos\alpha$?

c. If $\frac{\tan\alpha + 5}{6} = -\frac{\tan\alpha - 4}{2}$, what is $\tan\alpha$?

d. $(\tan 45° \cdot \cot\alpha) + (\sec 60° \cdot \cot\alpha) = 12$ is given. Calculate $\cot\alpha$.

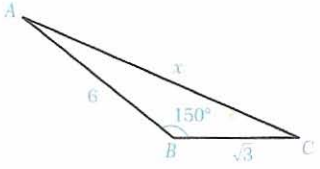
23. Each equation contains a trigonometric function. Find the value of the cofunction of this function in the given quadrant.

a. $\alpha \in (0°, 90°)$, $3(\tan\alpha - 4) = 2\tan\alpha - 9$

b. $\alpha \in (90°, 180°)$, $7(\sin\alpha - 1) = \frac{\sin\alpha - 10}{2}$

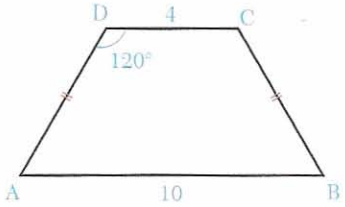
c. $\alpha \in (270°, 360°)$, $\frac{22 + \csc\alpha}{3 + 4\csc\alpha} = -4$

24. In the figure, $m(\angle ABC) = 150°$, $AB = 6$ and $BC = \sqrt{3}$. Calculate



a. $AC = x$. b. the area of ΔABC.

25. Find the perimeter P and area A of the trapezoid $ABCD$ in the figure.



26. Write each angle in degree-minute form. Give your answer rounded to two decimal places in the minutes place.

a. 48.5° b. 136.2° c. 213.75° d. 313.79°

27. Write each angle in decimal degrees. Give your answer rounded to two decimal places.

a. 121° 15′ b. 346° 50′

c. 198° 19′ d. 23° 56′ 12″

28. Find each trigonometric value rounded to four decimal places.

a. sin 23.4° b. cos 54.25°

c. tan 71.1° d. cot 63.55°

29. Find each trigonometric value rounded to four decimal places.

a. sin 121° 15′

b. cos 346° 50′

c. tan 131° 27′

d. cot 89° 49′

4 TRIGONOMETRIC THEOREMS AND FORMULAS

A. TRIGONOMETRIC THEOREMS



We now know how to calculate one or more angles or side lengths in a right triangle from given information about the triangle. However, there are also relations between the angles and sides of any triangle (not just right triangles). In this section we will study these general relations and use them to solve triangle problems.

1. The Law of Cosines

Theorem

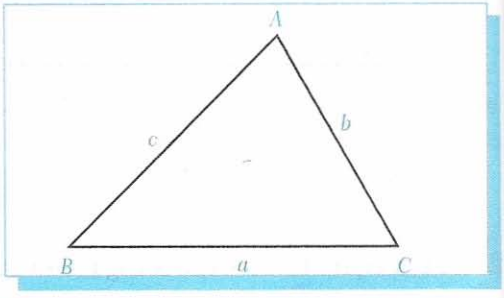
law of cosines

In a triangle ABC with side lengths a, b and c,

$a^2 = b^2 + c^2 - 2bc \cdot \cos A$

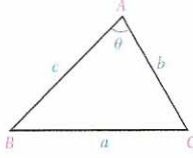
$b^2 = a^2 + c^2 - 2ac \cdot \cos B$

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$.



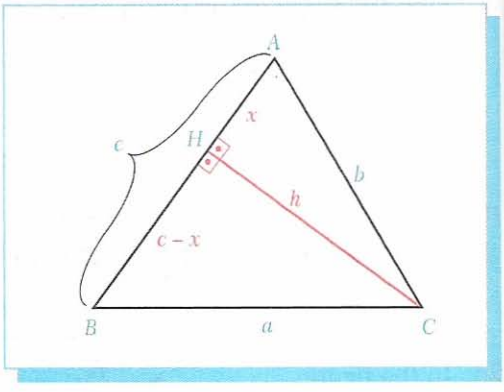
Proof

In a triangle ABC, a is the side opposite A, b is the side opposite B and c is the side opposite c.



a, b and c can also mean the lengths of sides a, b and c respectively.
We also write A to mean the angle at A, B to mean the angle at B and C to mean the angle at C: in the figure, $m(A) = \theta$.

First we draw CH as an altitude of the triangle. Point H now divides AB into two segments such that $AH = x$ and $HB = c - x$. Consequently there exist two right triangles: ΔCAH and ΔCHB.



Applying the Pythagorean Theorem to each right triangle gives us

$$h^2 + (c - x)^2 = a^2 \quad (1)$$

$$h^2 + x^2 = b^2. \quad (2)$$

We can subtract these side by side:

$$\begin{array}{rl} & h^2 + (c - x)^2 = a^2 \\ - & \quad h^2 + x^2 = b^2 \\ \hline h^2 + (c - x)^2 - h^2 - x^2 = & a^2 - b^2 \end{array}$$

$$c^2 - 2cx + x^2 - x^2 = a^2 - b^2.$$

$$\text{So } a^2 = b^2 + c^2 - 2cx. \quad (3)$$

In the right triangle CAH,

$\cos A = \dfrac{x}{b}$, i.e. $x = b \cdot \cos A$.

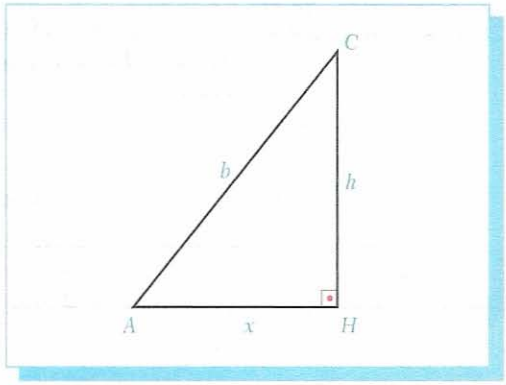
Substituting this value of x in (3) gives us

$$a^2 = b^2 + c^2 - 2cb \cdot \cos A.$$

We can rewrite this as

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A,$$

which is the required result.



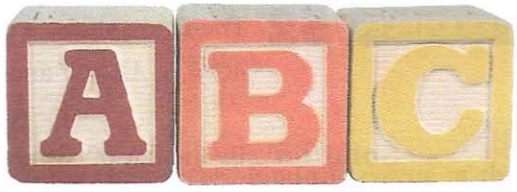
The proofs of the other two identities are similar. They are left as an exercise for you.

Notice that by rewriting the three parts of the law of cosines we obtain the relations

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

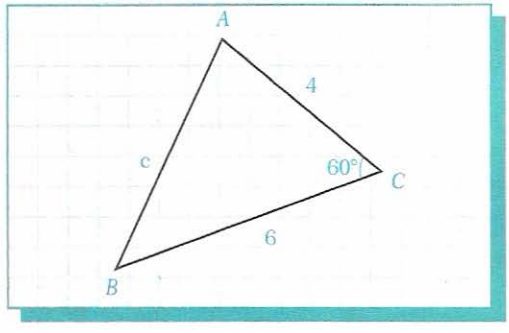


Note

By the law of cosines:

1. If two sides of a triangle and the angle between them are known, it is possible to calculate the length of the third side of the triangle.
2. If the three sides of a triangle are known, it is possible to calculate the trigonometric values of the angles in the triangle.

EXAMPLE 50 In the figure, ABC is a triangle with $AC = 4$, $BC = 6$ and $m(\angle ACB) = 60°$. Find AB.



Solution By the law of cosines,

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$

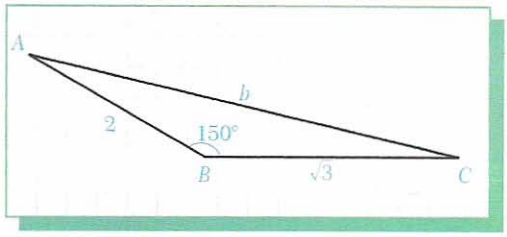
$c^2 = 6^2 + 4^2 - 2 \cdot 6 \cdot 4 \cdot \cos 60°$

$c^2 = 36 + 16 - \cancel{2} \cdot 6 \cdot 4 \cdot \dfrac{1}{\cancel{2}}$

$c^2 = 28$

$c = \sqrt{28} = 2\sqrt{7}$.

EXAMPLE 51 In the figure, ABC is a triangle with $AB = 2$, $BC = \sqrt{3}$ and $m(\angle ABC) = 150°$. Find b.



Solution By the law of cosines,

$b^2 = a^2 + c^2 - 2ac \cdot \cos B$

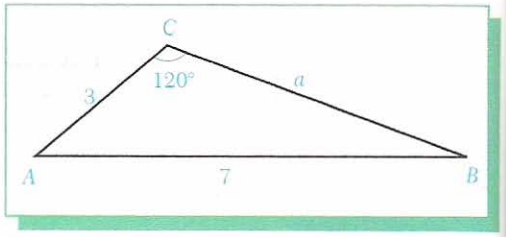
$b^2 = (\sqrt{3})^2 + 2^2 - 2 \cdot \sqrt{3} \cdot 2 \cdot \left(-\frac{\sqrt{3}}{2}\right)$ $\left(\cos 150° = -\frac{\sqrt{3}}{2}\right)$

$b^2 = 3 + 4 + 6$

$b^2 = 13$

$b = \sqrt{13}$.

EXAMPLE 52 The figure shows a triangle ABC with $AC = 3$, $AB = 7$ and $m(\angle ACB) = 120°$. Find a.



Solution By the law of cosines,

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$

$7^2 = a^2 + 3^2 - 2 \cdot a \cdot 3 \cdot \left(-\frac{1}{2}\right)$ $\left(\cos 120° = -\frac{1}{2}\right)$

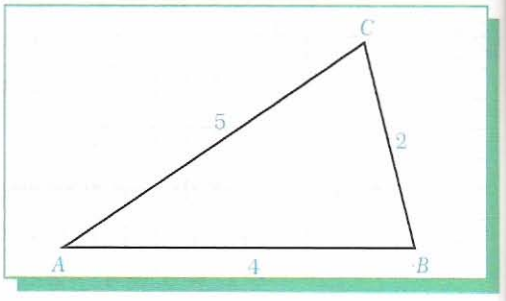
$49 = a^2 + 9 + 3a;$ $(a-5)(a+8) = 0;$

$a^2 + 3a - 40 = 0;$ $a = 5$, $a = -8.$

(–5) (8)

Since the length of a side cannot be negative, $a = 5$.

EXAMPLE 53 Find $\cos A$ in the triangle in the figure.



Solution By the law of cosines,

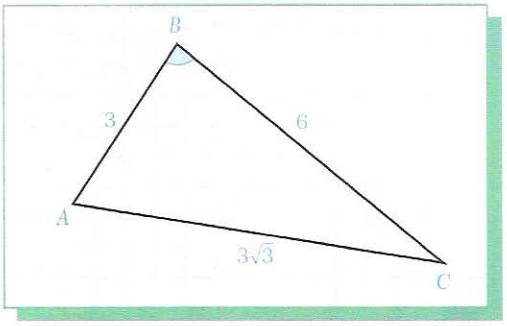
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{5^2 + 4^2 - 2^2}{2 \cdot 5 \cdot 4}$$

$$= \frac{25 + 16 - 4}{40}$$

$$= \frac{37}{40}.$$

EXAMPLE 54 Find the measure of angle B in the figure.



Solution By the law of cosines,

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{3^2 + 6^2 - (3\sqrt{3})^2}{2 \cdot 3 \cdot 6}$$

$$= \frac{9 + 36 - 27}{36}$$

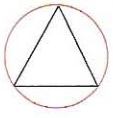
$$= \frac{18}{36} = \frac{1}{2}.$$

So $\cos B = \frac{1}{2}$. We know that $m(\angle B) < 90°$ and $\cos 60° = \frac{1}{2}$. Therefore $m(\angle B) = 60°$.

Check Yourself 15

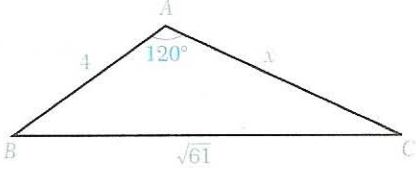
1. The side lengths of a triangle are 4, 5 and 6 units respectively. Find the cosine of the smallest angle in the triangle.

Inscribed and circumscribed:



The black line shows an inscribed triangle. The red line shows a circumscribed circle.

2. ABC is a triangle with $AB = 4$, $BC = \sqrt{61}$ and $m(\angle BAC) = 120°$. Find AC.



Answers

1. 0.75 2. 5

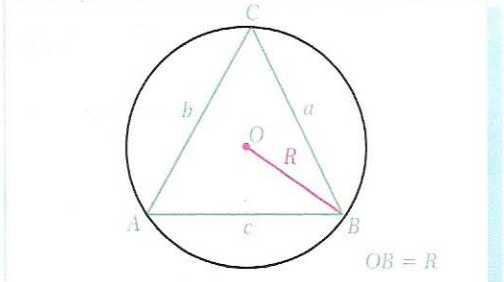
2. The Law of Sines

Theorem

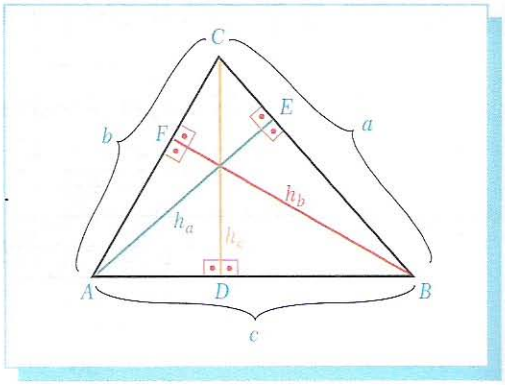
law of sines

Let ABC be a triangle with side lengths a, b and c which is inscribed in a circle with radius R.

Then $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.



Proof Look at the figure.



In the right triangle CDA,

$$\sin A = \frac{h_c}{b}, \text{ i.e. } h_c = b \cdot \sin A. \quad (1)$$

In the right triangle CDB,

$$\sin B = \frac{h_c}{a}, \text{ i.e. } h_c = a \cdot \sin B. \quad (2)$$

From (1) and (2)
we obtain $b \cdot \sin A = a \cdot \sin B$, i.e.

$$\frac{a}{\sin A} = \frac{b}{\sin B}. \quad (I)$$

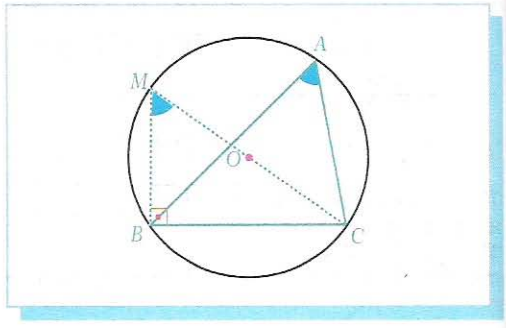
Similarly, in the right triangle BFA we have $\sin A = \frac{h_b}{c}$, i.e. $h_b = c \cdot \sin A$. (3)

Finally, in the right triangle BFC, $\sin C = \frac{h_b}{a}$ which gives $h_b = a \cdot \sin C$. (4)

From (3) and (4) we obtain $c \cdot \sin A = a \cdot \sin C$, i.e. $\frac{a}{\sin A} = \frac{c}{\sin C}$. (II)

From (I) and (II) we obtain $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. (Result 1)

To show the relation with the radius R of the circumscribed circle, we can use the figure opposite.



M is a point on the circle such that MC is the diameter of the circumscribed circle and $m(\angle MBC) = 90°$. Since $\angle BAC$ and $\angle BMC$ are circumscribed angles having the same arc $\overset{\frown}{BC}$ on the circle, $m(\angle BAC) = m(\angle BMC)$.

Since $m(\angle M) = m(\angle A)$ we have $\sin M = \sin A$.

In the right triangle MBC,

$$\sin M = \frac{a}{MC} = \frac{a}{2R}, \text{ i.e. } \sin A = \frac{a}{2R}.$$

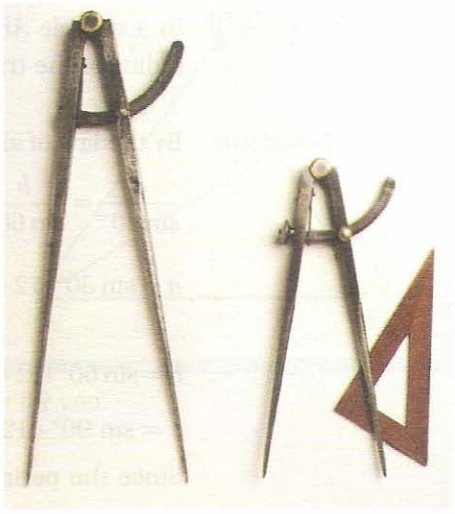
So $\frac{a}{\sin A} = 2R$. (Result 2)

Combining Result 1 and Result 2 gives us $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.

Note

Let ABC be an inscribed triangle. By the law of sines:

1. If the measures of the three angles and the radius of circumscribed circle are known, it is possible to find the side lengths of the triangle.
2. If the radius of the circumscribed circle and the side lengths of the triangle are known, it is possible to calculate the trigonometric values of the angles of the triangle.



EXAMPLE 55 In a triangle ABC, $m(\angle A) = 30°$ and the length of side a is 8 cm. Find the area of the circumscribed circle of ΔABC.

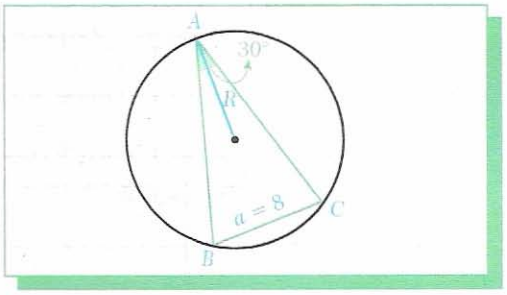
Solution By the law of sines,

$$\frac{a}{\sin A} = 2R \text{ so } \frac{8}{\sin 30°} = 2R.$$

So $\dfrac{8}{\frac{1}{2}} = 2R$, i.e. $R = 8$ cm.

We can calculate the area A of a circle with the formula $A = \pi R^2$.

So the area is $\pi \cdot 8^2 = 64\pi$ cm^2.

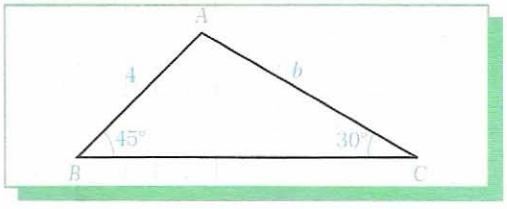


EXAMPLE 56 In a triangle ABC, $m(\angle B) = 45°$, $m(\angle C) = 30°$ and the length of side c is 4 cm. Find the length of side b.

Solution By the law of sines,

$$\frac{c}{\sin C} = \frac{b}{\sin B}, \text{ i.e. } \frac{4}{\sin 30°} = \frac{b}{\sin 45°}.$$

So $\dfrac{4}{\frac{1}{2}} = \dfrac{b}{\frac{\sqrt{2}}{2}}$ and $b = 4\sqrt{2}$ cm.



EXAMPLE 57 In a triangle ABC, the measures of the interior angles are 30°, 60° and 90° respectively. The radius of the triangle's circumscribed circle is 6 cm. Find the perimeter of the triangle ABC.

Solution By the law of sines,

$\frac{a}{\sin 30°} = \frac{b}{\sin 60°} = \frac{c}{\sin 90°} = 2 \cdot 6 = 12$. This gives

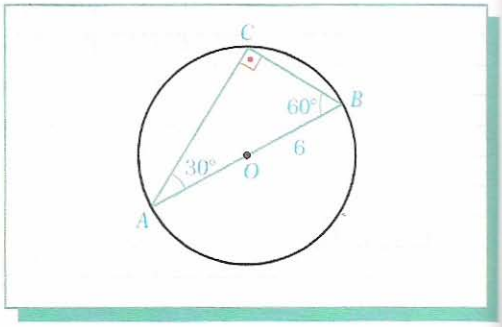
$a = \sin 30° \cdot 12 = \frac{1}{2} \cdot 12 = 6$ cm,

$b = \sin 60° \cdot 12 = \frac{\sqrt{3}}{2} \cdot 12 = 6\sqrt{3}$ cm and

$c = \sin 90° \cdot 12 = 12$ cm.

Since the perimeter $P(\Delta ABC) = a + b + c$ we can write

$P(\Delta ABC) = 6 + 6\sqrt{3} + 12 = 18 + 6\sqrt{3}$ cm.



Check Yourself 16

1. In a triangle ABC, $a = 5$ cm and the radius of the circumscribed circle is 5 cm. Find $m(\angle A)$.
2. In a triangle ABC, $m(\angle A) = 30°$, $m(\angle B) = 135°$ and $b = 4\sqrt{2}$ cm. Find the length of side a.

Answers

1. 30° or 150° 2. 4 cm

3. Formulas for the Area of a Triangle

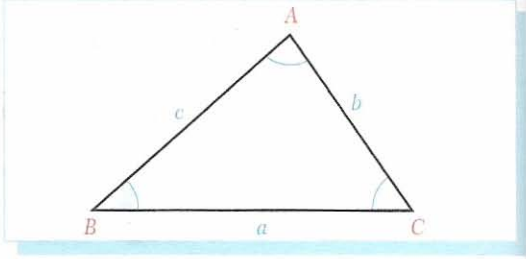
Theorem

In a triangle ABC with sides a, b and c, the following properties hold:

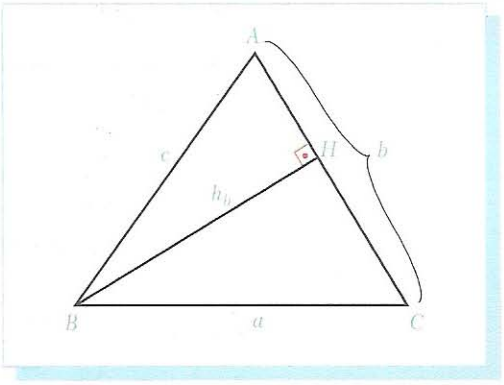
$A(\Delta ABC) = \frac{1}{2} \cdot a \cdot b \cdot \sin C$

$A(\Delta ABC) = \frac{1}{2} \cdot a \cdot c \cdot \sin B$

$A(\Delta ABC) = \frac{1}{2} \cdot b \cdot c \cdot \sin A.$



Proof Look at the figure. By the general rule for the area of a triangle, $A(\Delta ABC) = \frac{1}{2} \cdot b \cdot h_b$. (1)



Moreover, in the triangle ABH, $\sin A = \frac{h_b}{c}$ whic gives $h_b = c \cdot \sin A$. (2)

If we substitute (2) into (1) we obtain $A(\Delta ABC) = \frac{1}{2} \cdot b \cdot c \cdot \sin A$ which is one of the results to be proven.

The other proofs are similar and are left as an exercise for you.

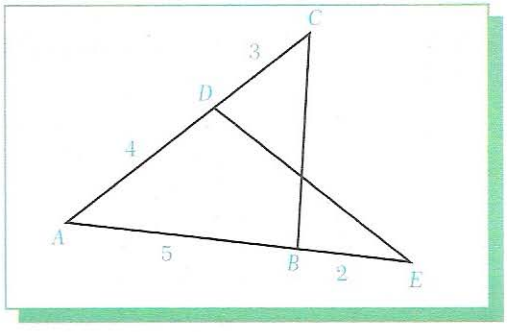
EXAMPLE 58 In a triangle ABC, $A(\Delta ABC) = 3\sqrt{3}$ cm^2, $b = 3$ cm and $c = 4$ cm. Find all the possible measures of angle A.

Solution $A(\Delta ABC) = \frac{1}{2} \cdot b \cdot c \cdot \sin A$

$3\sqrt{3} = \frac{1}{2} \cdot 3 \cdot 4 \cdot \sin A$, i.e. $\sin A = \frac{\sqrt{3}}{2}$

We know $\sin 60° = \sin 120° = \frac{\sqrt{3}}{2}$. So $m(\angle A) = 60°$ or $m(\angle A) = 120°$.

EXAMPLE 59 In the figure, $A(\Delta ABC) = 5$ cm^2. Find $A(\Delta ADE)$.



Solution $A(\Delta ABC) = \frac{1}{2} \cdot AB \cdot AC \cdot \sin A$

$5 = \frac{1}{2} \cdot 5 \cdot 7 \cdot \sin A$, i.e. $\sin A = \frac{2}{7}$

Since angle A is common to both triangles we can use the value of sin A for both triangles:

$A(\Delta ADE) = \frac{1}{2} \cdot AD \cdot AE \cdot \sin A$

$A(\Delta ADE) = \frac{1}{2} \cdot 4 \cdot 7 \cdot \frac{2}{7} = 4$ cm^2.

Theorem Heron's Formula

Let ABC be a triangle with sides a, b and c and perimeter $a + b + c = 2u$. Then $A(\Delta ABC) = \sqrt{u(u-a)(u-b)(u-c)}$.

Proof $\sin^2 A + \cos^2 A = 1$ so $\sin^2 A = 1 - \cos^2 A$. (1)

By the law of cosines, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$. Substituting this in (1) gives



$$\sin^2 A = 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2 = \frac{(2bc)^2 - (b^2 + c^2 - a^2)^2}{4b^2c^2}$$

$$= \frac{(2bc - b^2 - c^2 + a^2)(2bc + b^2 + c^2 - a^2)}{4b^2c^2}$$

$$= \frac{\left[a^2 - (b^2 - 2bc + c^2)\right]\left[(b^2 + 2bc + c^2) - a^2\right]}{4b^2c^2}$$

$$= \frac{\left[a^2 - (b-c)^2\right]\left[(b+c)^2 - a^2\right]}{4b^2c^2}$$

$$= \frac{(a+c-b)(a+b-c)(b+c-a)(a+b+c)}{4b^2c^2}.$$

Let us write each factor of the numerator in terms of u as follows:

$$\begin{array}{r} a + b + c = 2u \\ + \quad -2a = -2a \\ \hline b + c - a = 2(u-a) \end{array} \qquad \begin{array}{r} a + b + c = 2u \\ + \quad -2c = -2c \\ \hline a + b - c = 2(u-a) \end{array} \qquad \begin{array}{r} a + b + c = 2u \\ + \quad -2b = -2b \\ \hline a + c - b = 2(u-b). \end{array}$$

Now we have $\sin^2 A = \dfrac{2(u-a)\cdot 2(u-c)\cdot 2(u-b)\cdot 2u}{4b^2c^2}$, i.e.

$$= \frac{4u(u-a)(u-b)(u-c)}{b^2c^2}$$

$$= \frac{2}{bc}\sqrt{u(u-a)(u-b)(u-c)}.$$

If we use substitute this result in the area equation $A(\Delta ABC) = \dfrac{1}{2}\cdot b \cdot c \cdot \sin A$ we get

$$A(\Delta ABC) = \frac{1}{2}\cdot b \cdot c \cdot \frac{2}{bc}\sqrt{u(u-a)(u-b)(u-c)}.$$

So $A(\Delta ABC) = \sqrt{u(u-a)(u-b)(u-c)}$.

EXAMPLE 60 In a triangle ABC, $a = 9$ cm, $b = 10$ cm and $c = 11$ cm. Find $A(\Delta ABC)$.

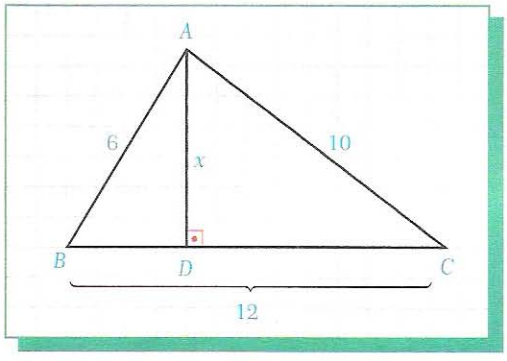
Solution By Heron's Formula, let $2u = a + b + c$ so $u = \dfrac{9+10+11}{2} = 15$.

Then $A(\Delta ABC) = \sqrt{u(u-a)(u-b)(u-c)}$

$= \sqrt{15(15-9)(15-10)(15-11)}$

$= \sqrt{15 \cdot 6 \cdot 5 \cdot 4} = 30\sqrt{2}$ cm^2.

EXAMPLE 61 In the figure, $AB = 6$ cm, $AC = 10$ cm and $BC = 12$ cm. Find $AD = x$.



Solution From Heron's Formula,

$2u = a + b + c$ so $u = \dfrac{12+10+6}{2} = 14$.

So $A(\Delta ABC) = \sqrt{u(u-a)(u-b)(u-c)}$

$= \sqrt{14(14-12)(14-10)(14-6)}$

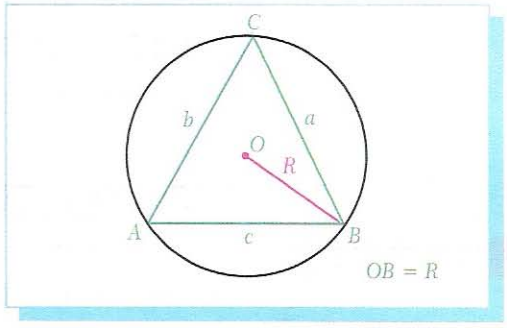
$= \sqrt{14 \cdot 2 \cdot 4 \cdot 8} = 8\sqrt{14}$ cm^2.

But $A(\Delta ABC) = \dfrac{1}{2} \cdot AD \cdot BC$, which gives us $8\sqrt{14} = \dfrac{1}{2} \cdot x \cdot 12$, i.e. $AD = x = \dfrac{4\sqrt{14}}{3}$ cm.

Theorem area of an inscribed triangle

Let ABC be a triangle with sides a, b, c which is inscribed in a circle with radius R. Then

$A(\Delta ABC) = \dfrac{a \cdot b \cdot c}{4R}$.



Proof $A(\Delta ABC) = \dfrac{1}{2} \cdot b \cdot c \cdot \sin A$. By the law of sines,

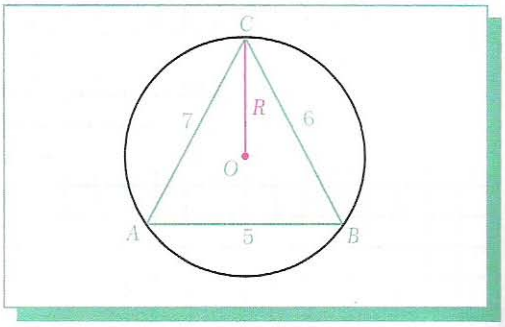
$\dfrac{a}{\sin A} = 2R$ which gives us $\sin A = \dfrac{a}{2R}$.

If we substitute $\dfrac{R}{2R}$ for $\sin A$ in the equation for the area, we obtain

$A(\Delta ABC) = \dfrac{1}{2} \cdot b \cdot c \cdot \dfrac{a}{2R} = \dfrac{a \cdot b \cdot c}{4R}$.

EXAMPLE 62 The side lengths of a triangle are 5 cm, 6 cm and 7 cm. Find the radius of the triangle's circumscribed circle.

Solution From Heron's Formula,



$2u = a + b + c$ so $u = \dfrac{5+6+7}{2} = 9$ and

$$A(\Delta ABC) = \sqrt{u(u-a)(u-b)(u-c)}$$

$$= \sqrt{9(9-5)(9-6)(9-7)}$$

$$= \sqrt{9 \cdot 4 \cdot 3 \cdot 2} = 6\sqrt{6} \text{ cm}^2.$$

However, since $A(\Delta ABC) = \dfrac{a \cdot b \cdot c}{4R}$ we have $6\sqrt{6} = \dfrac{5 \cdot 6 \cdot 7}{4R} = \dfrac{210}{4R}$.

Solving for R gives $R = \dfrac{35}{4\sqrt{6}}$ cm.

Check Yourself 17

1. In a triangle ABC, $AB = 8$ cm, $BC = 12$ cm and $m(\angle ABC) = 30°$. Find the area of the triangle.
2. A triangle has side lengths of 14 units, 16 units and 20 units respectively. Find its area.
3. In a triangle ABC, $A(\Delta ABC) = 18$ cm^2 and $a \cdot b \cdot c = 72$. Find the radius of circumscribed circle.

Answers

1. 24 cm^2 2. $15\sqrt{55}$ square units 3. 1 cm

4. Further Theorems (Optional)

Theorem

Let ABC be a triangle with sides a, b and c such that $A(\Delta ABC) = S$. Then

$$a = \sqrt{\frac{2S \cdot \sin A}{\sin B \cdot \sin C}}, \qquad b = \sqrt{\frac{2S \cdot \sin B}{\sin A \cdot \sin C}}, \qquad c = \sqrt{\frac{2S \cdot \sin C}{\sin A \cdot \sin B}}.$$

EXAMPLE 63 The interior angles of a triangle ABC measure 30°, 60° and 90° respectively. Given that $A(\Delta ABC) = S = 2\sqrt{3}$ cm^2, find the length of each side.

Solution Let us use the theorem.

$$a = \sqrt{\frac{2S \cdot \sin A}{\sin B \cdot \sin C}} = \sqrt{\frac{2 \cdot 2\sqrt{3} \cdot \sin 30°}{\sin 60° \cdot \sin 90°}} = \sqrt{\frac{2 \cdot 2\sqrt{3} \cdot \frac{1}{2}}{\frac{\sqrt{3}}{2} \cdot 1}} = 2 \text{ cm}$$

$$b = \sqrt{\frac{2S \cdot \sin B}{\sin A \cdot \sin C}} = \sqrt{\frac{2 \cdot 2\sqrt{3} \cdot \sin 60°}{\sin 30° \cdot \sin 90°}} = \sqrt{\frac{2 \cdot 2\sqrt{3} \cdot \frac{\sqrt{3}}{2}}{\frac{1}{2} \cdot 1}} = 2\sqrt{3} \text{ cm}$$

$$c = \sqrt{\frac{2S \cdot \sin C}{\sin A \cdot \sin B}} = \sqrt{\frac{2 \cdot 2\sqrt{3} \cdot \sin 90°}{\sin 30° \cdot \sin 60°}} = \sqrt{\frac{2 \cdot 2\sqrt{3} \cdot 1}{\frac{1}{2} \cdot \frac{\sqrt{3}}{2}}} = 4 \text{ cm}$$

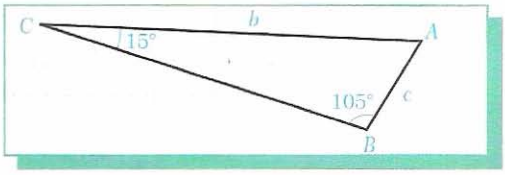
Theorem

law of tangents

In a triangle ABC with sides a, b and c,

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}, \qquad \frac{a+c}{a-c} = \frac{\tan\left(\frac{A+C}{2}\right)}{\tan\left(\frac{A-C}{2}\right)}, \qquad \frac{b+c}{b-c} = \frac{\tan\left(\frac{B+C}{2}\right)}{\tan\left(\frac{B-C}{2}\right)}.$$

EXAMPLE 64 In the figure, $m(\angle B) = 105°$, $m(\angle C) = 15°$ and $b + c = 3 + \sqrt{3}$. Find the lengths of sides b and c.



Solution By the law of tangents,

$$\frac{b+c}{b-c} = \frac{\tan\left(\frac{B+C}{2}\right)}{\tan\left(\frac{B-C}{2}\right)}. \text{ So } \frac{3+\sqrt{3}}{b-c} = \frac{\tan\left(\frac{105°+15°}{2}\right)}{\tan\left(\frac{105°-15°}{2}\right)}$$

$$= \frac{\tan(60°)}{\tan(45°)} = \frac{\sqrt{3}}{1}$$

$$= \sqrt{3}$$

$$b - c = \frac{3+\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}+3}{\sqrt{3}}$$

$$= \sqrt{3}+1.$$

If we solve the system $\begin{cases} b+c = 3+\sqrt{3} \\ b-c = \sqrt{3}+1 \end{cases}$, we find $b = 2\sqrt{3}$ and $c = 1$.

Check Yourself 18

1. In a triangle ABC, $a = 8$ cm, $A(\Delta ABC) = 16$ cm^2 and $\sin B \cdot \sin C = \frac{1}{4}$. Find $m(\angle A)$.
2. In a triangle ABC, $b = 11$ cm, $c = 5$ cm, $m(\angle B) = 78°$ and $m(\angle C) = 42°$. Find the value of tan 18°.

Answers

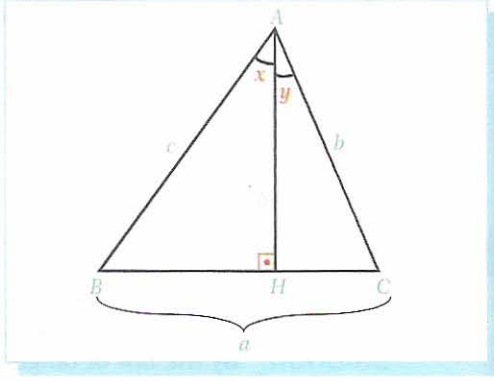
1. 30° or 150° 2. $\frac{3\sqrt{3}}{8}$

B. TRIGONOMETRIC FORMULAS

1. Sum and Difference Formulas

In this section we will learn the relations between the sum or difference of two angles and their trigonometric ratios. We will prove these relations using the trigonometric identities we have studied.

a. sin(x±y)



In the figure,

$A(\Delta ABC) = A(\Delta ABH) + A(\Delta ACH)$. By the formula for the area of a triangle,

$$\frac{1}{2} \cdot b \cdot c \cdot \sin A = \frac{1}{2} \cdot c \cdot h \cdot \sin x + \frac{1}{2} \cdot b \cdot h \cdot \sin y,$$

which we can rewrite as

$(b \cdot c \cdot \sin A) = (c \cdot h \cdot \sin x) + (b \cdot h \cdot \sin y)$.

If we divide both sides by $b \cdot c$ we get

$$\sin A = \frac{h}{b} \cdot \sin x + \frac{h}{c} \cdot \sin y. \quad (1)$$

In the triangle, $m(A) = x + y$, $\cos y = \frac{h}{b}$ and $\cos x = \frac{h}{c}$.

If we substitute these in (1) we obtain

$$\boxed{\sin (x + y) = [\sin x \cdot \cos y] + [\cos x \cdot \sin y]}.$$

If we replace y with $-y$ in this equation we get

$\sin (x + (-y)) = [\sin x \cdot \cos (-y)] + [\cos x \cdot \sin (-y)]$.

Since $\cos (-y) = \cos y$ and $\sin (-y) = -\sin y$ we have

$$\boxed{\sin (x - y) = [\sin x \cdot \cos y] - [\cos x \cdot \sin y]}.$$

EXAMPLE 65 Calculate sin 75°.

Solution We can write 75° as the sum or difference of two easier angles.

For example, we know the values of the trigonometric functions of 45° and 30°, so we can write 75° = 45°+ 30°.

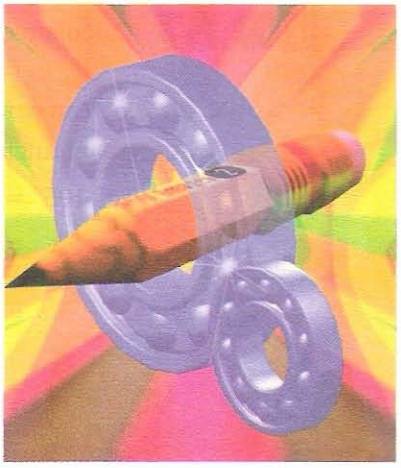
So $\sin 75° = \sin (45° + 30°)$.

By the sine of the sum of two angles,

$$\sin(45° + 30°) = (\sin 45° \cdot \cos 30°) + (\cos 45° \cdot \sin 30°)$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}. \quad \text{So } \sin 75° = \frac{\sqrt{6} + \sqrt{2}}{4}.$$



EXAMPLE 66 Calculate sin 15°.

Solution We can write 15° as 45° – 30°, 60° – 45° or any other suitable combination.

Let us choose 60° – 45°.

$$\sin 15° = \sin (60° - 45°)$$

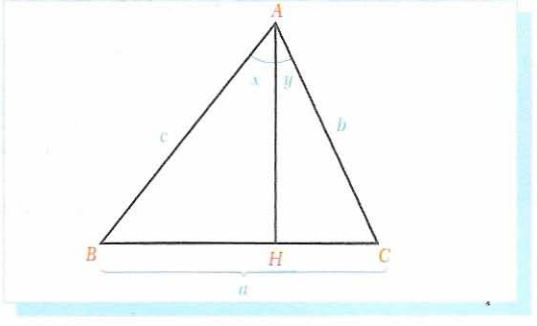
$$= (\sin 60° \cdot \cos 45°) - (\cos 60° \cdot \sin 45°)$$

$$= \left(\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2} \cdot \frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

b. cos(x±y)

To find cos $(x + y)$ we will use the formula for the sine of the difference of two angles obtained in the previous section.



$\cos \alpha = \sin(90° - \alpha)$

$$\cos (x + y) = \sin (90° - (x + y))$$

$$= \sin (90° - x - y)$$

$$= \sin ((90° - x) - y) \text{ (regrouping)}$$

$$= \sin ((90° - x) - y)$$

$$= \sin [(90° - x) \cdot \cos y] - [\cos (90° - x) \cdot \sin y].$$

Since $\sin (90° - x) = \cos x$ and $\cos (90° - x) = \sin x$,

$$\cos (x + y) = [\cos x \cdot \cos y] - [\sin x \cdot \sin y].$$

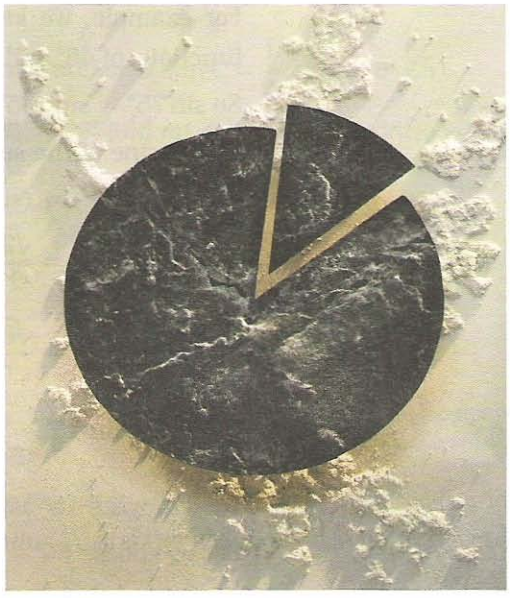
If we replace y with $-y$ in this equation we get

$\cos (x + (-y)) = [\cos x \cdot \cos (-y)] - [\sin x \cdot \sin (-y)]$.

Since $\cos (-y) = \cos y$ and $\sin (-y) = -\sin y$ we have

$$\cos (x - y) = [\cos x \cdot \cos y] + [\sin x \cdot \sin y].$$

EXAMPLE 67 Calculate $\cos 105°$.



Solution We can write $105°$ as the sum or difference of two easier angles. Let us choose $105° = 60° + 45°$.

$$\cos 105° = \cos (60° + 45°)$$

$$= (\cos 60° \cdot \cos 45°) - (\sin 60° \cdot \sin 45°)$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

EXAMPLE 68 Show that $\cos (60° + 30°) \neq \cos 60° + \cos 30°$.

Solution 1 We have $60° + 30° = 90°$ and we know $\cos 90° = 0$. On the other hand,

$\cos 60° + \cos 30° = \frac{1}{2} + \frac{\sqrt{3}}{2} \neq 0$. Therefore, $\cos (60° + 30°) \neq \cos 60° + \cos 30°$.

Solution 2 We know that $\cos (x + y) = (\cos x \cdot \cos y) - (\sin x \cdot \sin y)$.

So $\cos (60° + 30°) = (\cos 60° \cdot \cos 30°) - (\sin 60° \cdot \sin 30°)$, i.e.

$$\cos (60° + 30°) = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = 0.$$

However, $\cos 60° + \cos 30° = \frac{1}{2} + \frac{\sqrt{3}}{2} \neq 0$. So $\cos (60° + 30°) \neq \cos 60° + \cos 30°$.

c. tan(x±y)

We know that $\tan\theta = \frac{\sin\theta}{\cos\theta}$. So $\tan (x + y) = \frac{\sin (x + y)}{\cos (x + y)} = \frac{(\sin x \cdot \cos y) + (\cos x \cdot \sin y)}{(\cos x \cdot \cos y) - (\sin x \cdot \sin y)}$ by our previous results.

If we divide the numerator and denominator by $\cos x \cdot \cos y$ and simplify, we get

$$\tan(x+y)=\frac{\dfrac{(\sin x\cdot\cos y)+(\cos x\cdot\sin y)}{\cos x\cdot\cos y}}{\dfrac{(\cos x\cdot\cos y)-(\sin x\cdot\sin y)}{\cos x\cdot\cos y}}$$

$$=\frac{\dfrac{\sin x\cdot\cos y}{\cos x\cdot\cos y}+\dfrac{\cos x\cdot\sin y}{\cos x\cdot\cos y}}{\dfrac{\cos x\cdot\cos y}{\cos x\cdot\cos y}-\dfrac{\sin x\cdot\sin y}{\cos x\cdot\cos y}}$$

$$=\frac{\dfrac{\sin x}{\cos x}+\dfrac{\sin y}{\cos y}}{1-\dfrac{\sin x}{\cos x}\cdot\dfrac{\sin y}{\cos y}}.$$ This gives us

$$\tan(x+y)=\frac{\tan x+\tan y}{1-(\tan x\cdot\tan y)}.$$

If we replace y with $-y$ in this equation we get $\tan(x+(-y))=\dfrac{\tan x+\tan(-y)}{1-(\tan x\cdot\tan(-y))}$.

Since $\tan(-y) = -\tan y$ we have

$$\tan(x-y)=\frac{\tan x-\tan y}{1+(\tan x\cdot\tan y)}.$$

EXAMPLE 69 Verify that $\tan 210^\circ=\dfrac{\sqrt{3}}{3}$.

Solution $210^\circ = 180^\circ + 30^\circ$

$$\tan 210^\circ=\tan(180^\circ+30^\circ)=\frac{\tan 180^\circ+\tan 30^\circ}{1-(\tan 180^\circ\cdot\tan 30^\circ)}$$ (by the formula above)

$$=\frac{0+\dfrac{\sqrt{3}}{3}}{1-0\cdot\dfrac{\sqrt{3}}{3}}=\frac{\sqrt{3}}{3}$$

EXAMPLE 70 Find $\tan(x-y)$ using $\tan x=\dfrac{5}{2}$ and $\tan y=\dfrac{1}{4}$.

Solution $\tan(x-y)=\dfrac{\tan x-\tan y}{1+(\tan x\cdot\tan y)}=\dfrac{\frac{5}{2}-\frac{1}{4}}{1+\frac{5}{2}\cdot\frac{1}{4}}=\dfrac{\frac{9}{4}}{\frac{13}{8}}$

$$=\frac{18}{13}$$

d. cot(x±y)

$$\cot(x+y)=\frac{1}{\tan(x+y)}=\frac{\cos(x+y)}{\sin(x+y)}=\frac{(\cos x\cdot\cos y)-(\sin x\cdot\sin y)}{(\sin x\cdot\cos y)+(\cos x\cdot\sin y)}$$

Let us divide the numerator and denominator by $\sin x \cdot \sin y$ and simplify:

$$\cot(x+y)=\frac{\dfrac{(\cos x\cdot\cos y)-(\sin x\cdot\sin y)}{\sin x\cdot\sin y}}{\dfrac{(\sin x\cdot\cos y)+(\cos x\cdot\sin y)}{\sin x\cdot\sin y}}$$

$$=\frac{\dfrac{\cos x\cdot\cos y}{\sin x\cdot\sin y}-\dfrac{\sin x\cdot\sin y}{\sin x\cdot\sin y}}{\dfrac{\sin x\cdot\cos y}{\sin x\cdot\sin y}+\dfrac{\cos x\cdot\sin y}{\sin x\cdot\sin y}}$$

$$=\frac{\dfrac{\cos x}{\sin x}\cdot\dfrac{\cos y}{\sin y}-1}{\dfrac{\cos y}{\sin y}+\dfrac{\cos x}{\sin x}}.$$ So

$$\cot(x+y)=\frac{(\cot x\cdot\cot y)-1}{\cot y+\cot x}.$$

If we replace y with $-y$ in this equation we get $\cot(x+(-y))=\dfrac{\cot x\cdot\cot(-y)-1}{\cot(-y)+\cot x}$.

Since $\cot(-y) = -\cot y$ we have $\cot(x-y)=\dfrac{-(\cot x\cdot\cot y)-1}{-\cot y+\cot x}$, i.e.

$$\cot(x-y)=\frac{(\cot x\cdot\cot y)+1}{\cot y-\cot x}.$$

Note

We can also calculate these results by using the corresponding results for the tangent and the fact that $\cot\alpha=\dfrac{1}{\tan\alpha}$ (so $\cot(x\pm y)=\dfrac{1}{\tan(x\pm y)}$).

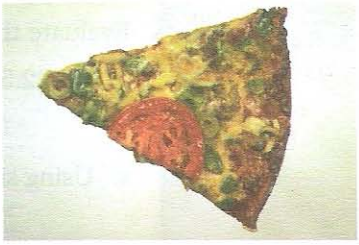
EXAMPLE 71 Calculate cot 75°.

Solution cos 75° = cot(45° + 30°)

$$=\frac{(\cot 45^\circ\cdot\cot 30^\circ)-1}{\cot 45^\circ+\cot 30^\circ}=\frac{(1\cdot\sqrt{3})-1}{1+\sqrt{3}}=\frac{\sqrt{3}-1}{1+\sqrt{3}}=\frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$=\frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2-(\sqrt{1})^2}=\frac{3-2\sqrt{3}+1}{2}=\frac{4-2\sqrt{3}}{2}=2-\sqrt{3}$$

Check Yourself 19



1. Calculate cos 15° and sin 105°.
2. Calculate tan 195° and cot 285°.
3. Verify the results.
 a. $\sin 135° = \frac{\sqrt{2}}{2}$ b. $\cos 300° = \frac{1}{2}$
4. Calculate tan 15° + cot 15°.
5. $\cot x = -1$ and $\cot y = -\frac{4}{3}$ are given. Find $\cot(x - y)$.

Answers

1. $\cos 15° = \sin 105° = \frac{\sqrt{2}+\sqrt{6}}{4}$ 2. $\tan 195° = 2 - \sqrt{3}$, $\cot 285° = \sqrt{3} - 2$ 4. 4 5. –7

2. Double-Angle and Half-Angle Formulas

We now know formulas to calculate trigonometric ratios such as $\sin(x + y)$, $\cos(x + y)$, $\tan(x + y)$ and $\cot(x + y)$. In this section we will consider the special case $x = y$ and find formulas for the trigonometric ratios $\sin 2x$, $\cos 2x$, $\tan 2x$ and $\cot 2x$. These formulas are called the **double-angle formulas**.

a. sin 2x

We know that $\sin(x + y) = (\sin x \cdot \cos y) + (\cos x \cdot \sin y)$.

If $x = y$ this formula becomes

$\sin(x + x) = (\sin x \cdot \cos x) + (\cos x \cdot \sin x)$, i.e.

$$\sin 2x = 2\sin x \cdot \cos x .$$

We can also rewrite this as $\sin x \cdot \cos x = \frac{\sin 2x}{2}$.

EXAMPLE 72 Calculate sin 120° using the double-angle formula for sine.

Solution

$$\sin 120° = \sin(2 \cdot 60°)$$
$$= 2 \cdot \sin 60° \cdot \cos 60° \quad \text{(double-angle formula)}$$
$$= 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{3}}{2}$$



EXAMPLE 73 Evaluate the expressions.

a. $\sin 22.5° \cdot \cos 22.5° \cdot \cos 45°$

b. $6\sin\frac{\pi}{8}\cos\frac{\pi}{8}$

Solution a. Using $\sin x \cdot \cos x = \frac{\sin 2x}{2}$ gives us

$$\sin 22.5° \cdot \cos 22.5° \cdot \cos 45° = \frac{1}{2}\sin(2 \cdot 22.5°) \cdot \cos 45°$$

$$= \frac{1}{2}\sin 45° \cdot \cos 45°$$

$$= \frac{1}{2} \cdot \frac{1}{2}\sin(2 \cdot 45°)$$

$$= \frac{1}{4}\sin 90° = \frac{1}{4}.$$

b. $6\sin\frac{\pi}{8}\cos\frac{\pi}{8} = 3 \cdot (2\sin\frac{\pi}{8}\cos\frac{\pi}{8}) = 3 \cdot \sin(2 \cdot \frac{\pi}{8}) = 3 \cdot \sin\frac{\pi}{4} = 3 \cdot \frac{\sqrt{2}}{2}$

$$= \frac{3\sqrt{2}}{2}$$

EXAMPLE 74 Simplify $\frac{\cos 6x}{\cos 2x} - \frac{\sin 6x}{\sin 2x}$.

Solution $\frac{\cos 6x}{\cos 2x} - \frac{\sin 6x}{\sin 2x} = \frac{(\cos 6x \cdot \sin 2x) - (\cos 2x \cdot \sin 6x)}{\cos 2x \cdot \sin 2x} = \frac{(\sin 2x \cdot \cos 6x) - (\sin 6x \cdot \cos 2x)}{\sin 2x \cdot \cos 2x}$.

Using the double-angle formula gives us $\frac{\sin(2x - 6x)}{\sin 2x \cdot \cos 2x} = \frac{\sin(-4x)}{\cos 2x \cdot \sin 2x}$, and using the double-angle formula again gives us $\frac{-\sin 4x}{\frac{1}{2}\sin 4x} = -2$.

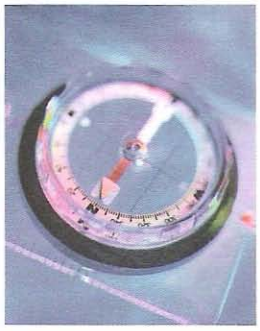
EXAMPLE 75 Simplify $\frac{4 \cdot \cos 50° \cdot \sin 50° \cdot \cos 100°}{\sin 200°}$.

Solution We can rewrite this as $\frac{2 \cdot (2 \cdot \cos 50° \cdot \sin 50°) \cdot \cos 100°}{\sin 200°}$ and use $\sin 100° = 2 \cdot \sin 50° \cdot \cos 50°$.

This gives us $\frac{2 \cdot \sin 100° \cdot \cos 100°}{\sin 200°}$ by the double-angle formula.

Using the double-angle formula again gives us $\frac{\sin 200°}{\sin 200°} = 1$.

EXAMPLE 76 Evaluate $\sin 20° \cdot \cos 40° \cdot \cos 80°$.



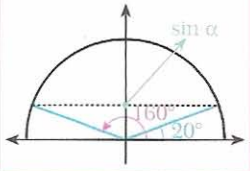
Solution

$$\sin 20° \cdot \cos 40° \cdot \cos 80° = \frac{(\sin 20° \cdot \cos 20°) \cdot \cos 40° \cdot \cos 80°}{\cos 20°}$$

$$= \frac{\frac{1}{2}(\sin 40° \cdot \cos 40°) \cdot \cos 80°}{\cos 20°}$$

$$= \frac{\frac{1}{4}\sin 80° \cdot \cos 80°}{\cos 20°} = \frac{\frac{1}{8}\sin 160°}{\cos 20°} = \frac{\frac{1}{8}\sin 20°}{\cos 20°} = \frac{1}{8}\tan 20°$$

$\sin 160° = \sin 20°$



b. cos 2x

We know that $\cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y$.

If $x = y$ this formula becomes

$\cos(x + x) = (\cos x \cdot \cos x) - (\sin x \cdot \sin x)$, i.e.

$$\cos 2x = \cos^2 x - \sin^2 x$$

We can also use the identities $\sin^2 x = 1 - \cos^2 x$ and $\cos^2 x = 1 - \sin^2 x$ to obtain two additional formulas:

$$\cos 2x = 2\cos^2 x - 1$$
$$\cos 2x = 1 - 2\sin^2 x$$

EXAMPLE 77 Calculate $\cos 2x$ given $\cos x = \frac{\sqrt{7}}{4}$ and $0 < x < \frac{\pi}{2}$.

Solution

$$\cos 2x = 2 \cdot \cos^2 x - 1 = 2 \cdot \left(\frac{\sqrt{7}}{4}\right)^2 - 1 = 2 \cdot \frac{7}{16} - 1$$

$$= -\frac{1}{8}$$

EXAMPLE 78 Simplify the expression $\frac{1 - \cos 2x}{1 + \cos 2x}$.

Solution

$$\frac{1 - \cos 2x}{1 + \cos 2x} = \frac{1 - (1 - 2\sin^2 x)}{1 + (2\cos^2 x - 1)} = \frac{1 - 1 + 2\sin^2 x}{1 + 2\cos^2 x - 1} = \frac{2\sin^2 x}{2\cos^2 x}$$

$$= \tan^2 x$$

EXAMPLE 79 Given $\cos 11° = t$, write $\sin 68°$ in terms of t.

Solution

$\sin 68° = \cos 22°$ (cofunctions)

$\cos 22° = \cos(2 \cdot 11°) = 2 \cdot \cos^2 11° - 1$ (double-angle formula)

$= \sin 68° = 2t^2 - 1$

c. tan 2x

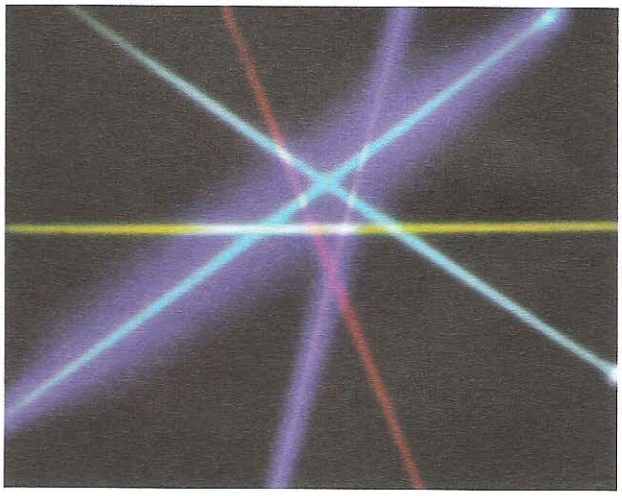
We know that $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$.

If $x = y$ this formula becomes

$\tan(x+x) = \dfrac{\tan x + \tan x}{1 - \tan x \cdot \tan x}$, i.e.

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}.$$

EXAMPLE 80 $\tan x = \dfrac{4}{3}$. Find $\tan 2x$.



Solution

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$= \frac{2 \cdot \frac{4}{3}}{1 - \left(\frac{4}{3}\right)^2} = \frac{\frac{8}{3}}{1 - \frac{16}{9}} = \frac{\frac{8}{3}}{-\frac{7}{9}}$$

$$= -\frac{24}{7}$$

d. cot 2x

We know that $\cot(x+y) = \dfrac{\cot x \cdot \cot y - 1}{\cot x + \cot y}$.

If $x = y$ this formula becomes

$\cot(x+x) = \dfrac{\cot x \cdot \cot x - 1}{\cot x + \cot x}$, i.e.

$$\cot 2x = \frac{\cot^2 x - 1}{2\cot x}.$$

EXAMPLE 81 Given that x is an acute angle, calculate $\cot 2x$ using $\tan x - \cot x = 2$.

Solution Since $\tan x = \dfrac{1}{\cot x}$ we have $\dfrac{1}{\cot x} - \cot x = 2$, i.e.

$$\frac{1-\cot^2 x}{\cot x} = \frac{2}{1}.$$

This gives us $1 - \cot^2 x = 2\cot x$, i.e. $\cot^2 x + 2\cot x - 1 = 0$.

If we apply the quadratic formula we get $\cot x = \dfrac{-2-\sqrt{6}}{2}$ or $\cot x = \dfrac{-2+\sqrt{6}}{2}$.

Since x is an acute angle, the cotangent value must be positive. Since $\sqrt{6} > 2$, $-2 + \sqrt{6}$ is positive and so $\cot x = \dfrac{-2+\sqrt{6}}{2}$.

e. Half-angle formulas

We have just seen that $\sin 2x$, $\cos 2x$, $\tan 2x$, and $\cot 2x$ can be expressed in terms of $\sin x$, $\cos x$, $\tan x$ and $\cot x$ respectively. In addition, we can apply the procedure in reverse order to express $\sin x$, $\cos x$, $\tan x$, and $\cot x$ in terms of $\sin 2x$, $\cos 2x$, $\tan 2x$ and $\cot 2x$ respectively. For this reason, the double-angle formulas are also called half-angle formulas.

By using the double-angle formula for the cosine function, we can obtain the half-angle formulas for the sine, tangent and cotangent functions as follows.

We know that $\cos 2x = 2\cos^2 x - 1$. If we replace x with $\dfrac{x}{2}$, then

$$\cos\left(2\cdot\frac{x}{2}\right) = 2\cos^2\frac{x}{2} - 1,$$

$$\cos x = 2\cos^2\frac{x}{2} - 1 \text{ i.e. } \boxed{\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}}. \quad (1)$$

Similarly, $\cos 2x = 1 - 2\sin^2 x$. If we replace x with $\dfrac{x}{2}$ then $\cos\left(2\cdot\dfrac{x}{2}\right) = 1 - \sin^2\dfrac{x}{2}$,

$$\cos x = 1 - \sin^2\frac{x}{2} \text{ i.e. } \boxed{\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}}. \quad (2)$$

Using (1) and (2) we can write

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\pm\sqrt{\frac{1-\cos x}{2}}}{\pm\sqrt{\frac{1+\cos x}{2}}}, \text{ i.e. } \boxed{\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}}.$$

Similarly,

$$\cot\frac{x}{2}=\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}=\frac{\pm\sqrt{\frac{1+\cos x}{2}}}{\pm\sqrt{\frac{1-\cos x}{2}}}, \text{ i.e.}$$

$$\cot\frac{x}{2}=\pm\sqrt{\frac{1+\cos x}{1-\cos x}}.$$



EXAMPLE 82 $\cos 2x=-\frac{1}{5}$ is given. Find $\cos x$ if x is in the first quadrant.

Solution $\cos 2x = 2\cos^2 x - 1$ so $\cos x=\pm\sqrt{\frac{1+\cos 2x}{2}}$.

Since x is in the first quadrant, the cosine is positive.

$$\text{So } \cos x=\sqrt{\frac{1+\left(-\frac{1}{5}\right)}{2}}$$

$$=\sqrt{\frac{2}{5}}.$$

EXAMPLE 83 Calculate $\sin 22.5^\circ$ using half-angle formulas.

Solution $\cos 2x = 1 - 2\sin^2 x$ so $\sin x=\pm\sqrt{\frac{1-\cos 2x}{2}}$. Since x is an acute angle, the sine is positive.

$$\text{So } \sin(22.5^\circ)=\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} \qquad \left(\cos 45^\circ=\frac{\sqrt{2}}{2}\right)$$

$$=\frac{\sqrt{2-\sqrt{2}}}{2}.$$

EXAMPLE 84 $\cos x = \frac{3}{5}$ is given. Find $\tan\frac{x}{2}$ if x is in the fourth quadrant.

Solution Since x is in the fourth quadrant, $270° < x < 360°$.

So $135° < \frac{x}{2} < 180°$. This means that $\frac{x}{2}$ is in the second quadrant, where the tangent function is negative.

By the half-angle formula, $\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$ and we take the negative value.

Hence, $\tan\frac{x}{2} = -\sqrt{\frac{1-\frac{3}{5}}{1+\frac{3}{5}}} = -\sqrt{\frac{\frac{2}{5}}{\frac{8}{5}}}$, i.e. $\tan\frac{x}{2} = -\sqrt{\frac{1}{4}} = -\frac{1}{2}$.

EXAMPLE 85 Find the values of sin 105° and cos 15° using half-angle formulas.

Solution $\sin 105° = \sin\frac{210°}{2} = \sqrt{\frac{1-\cos 210°}{2}}$ $\left(\cos 210° = -\frac{\sqrt{3}}{2}\right)$

Notice that we take the positive value in the half-angle formula because the sine function is positive in the second quadrant. So we have

$$\sin 105° = \sqrt{\frac{1-(-\frac{\sqrt{3}}{2})}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

Similarly,

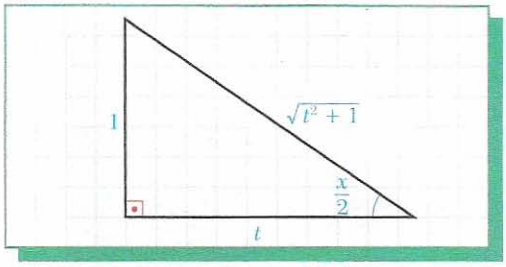
$$\cos 15° = \cos\frac{30°}{2} = \sqrt{\frac{1+\cos 30°}{2}}. \qquad \left(\cos 30° = \frac{\sqrt{3}}{2}\right)$$

We take the positive value because the cosine function is positive in the first quadrant. So

$$\cos 15° = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}}$$

$$= \frac{\sqrt{2+\sqrt{3}}}{2}.$$

EXAMPLE 86 Given $\cot\frac{x}{2}=t$, find $\sin x$ in terms of t.



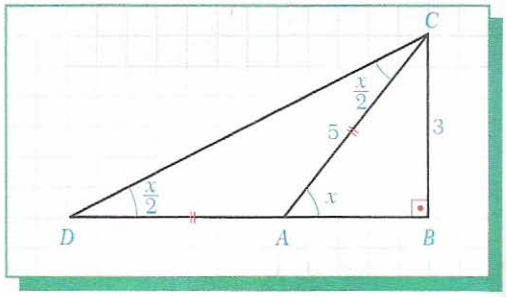
Solution We can show $\cot\frac{x}{2}=\frac{t}{1}$ in a right triangle, as shown opposite.

From the double-angle formula for the sine we have $\sin x = 2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}$.

So $\sin x = 2\cdot\frac{1}{\sqrt{t^2+1}}\cdot\frac{t}{\sqrt{t^2+1}}$

$$=\frac{2t}{t^2+1}.$$

EXAMPLE 87 Given $\sin x=\frac{3}{5}$, find $\cos\frac{x}{2}$ if x is an acute angle.



Solution We can solve this problem using the previous formulas. However, let us look at an alternative geometric solution.

Step 1: We sketch the triangle ABC as in the figure. We calculate $AB = 4$ using the Pythagorean Theorem.

Step 2: We extend side AB to create DB such that $AD = AC = 5$.

Step 3: We construct the right triangle DBC.

Since $AD = AC$, the triangle DAC is an isosceles triangle.

Hence, $m(\angle CDA) = m(\angle ACD)$ and $m(\angle CAB) = m(\angle CDA) + m(\angle ACD)$.

Therefore, $m(\angle CDB)=\frac{x}{2}$. Moreover, $BD = AD + AB = 5 + 4 = 9$.

By using the Pythagorean Theorem with the triangle CDB,

$$DC^2 = BC^2 + BD^2$$

$$= 3^2 + 9^2$$

$$DC = \sqrt{9+81} = \sqrt{90}$$

$$= 3\sqrt{10}.$$

So $\cos\frac{x}{2}=\frac{DB}{DC}=\frac{9}{3\sqrt{10}}$

$$=\frac{3\sqrt{10}}{10}.$$

Check Yourself 20

1. α is an acute angle such that $\sin\frac{\alpha}{2}=\frac{3}{5}$. Find $\sin\alpha$.

2. α is an acute angle such that $\cos\alpha=\frac{2}{5}$. Find $\cos 2\alpha$.

3. $\cos 2x\cdot\cos x\cdot\cos\frac{x}{2}\cdot\sin\frac{x}{2}=\frac{1}{24}$ is given. Find $\cos 8x$.

4. If $\tan 84° = t$, find $\cot 78°$ in terms of t.

5. Calculate $\tan 15°$.

Answers

1. $\frac{24}{25}$ 2. $-\frac{17}{25}$ 3. $\frac{7}{9}$ 4. $\frac{2t}{t^2-1}$ 5. $2-\sqrt{3}$

3. Reduction Formulas

We have already learned how to find trigonometric values by means of a reference angle. In addition, by using the sum and difference formulas we can derive new relations for the sum or difference of a variable angle and a quadrantal angle. For example, consider the value $\sin(\frac{\pi}{2}+y)$.

We know that $\sin(x+y)=(\sin x\cdot\cos y)+(\cos x\cdot\sin y)$. If we replace x with $\frac{\pi}{2}$, the formula becomes $\sin\left(\frac{\pi}{2}+y\right)=\left(\sin\frac{\pi}{2}\cdot\cos y\right)+\left(\cos\frac{\pi}{2}\cdot\sin y\right)$.

Evaluating the quadrantal angles gives us $\sin\left(\frac{\pi}{2}+y\right)=1\cdot\cos y+0\cdot\sin y$.

Hence, $\sin\left(\frac{\pi}{2}+y\right)=\cos y$. (1)

Similarly, we know that $\cos(x-y)=(\cos x\cdot\cos y)+(\sin x\cdot\sin y)$.

If we replace x with $\frac{3\pi}{2}$, the formula becomes $\cos\left(\frac{3\pi}{2}-y\right)=\left(\cos\frac{3\pi}{2}\cdot\cos y\right)+\left(\sin\frac{3\pi}{2}\cdot\sin y\right)$.

After evaluation we obtain $\cos\left(\frac{3\pi}{2}-y\right)=0\cdot\cos y+(-1)\cdot\sin y$.

Hence, $\cos\left(\frac{3\pi}{2}-y\right)=-\sin y$. (2)

As a third example, let us find a trigonometric equivalent of $\tan(-\alpha)$. Since $\tan(-\alpha)$ can be written as $\tan(2\pi - \alpha)$ or $\tan(0 - \alpha)$, we can apply the formula $\tan(x-y) = \dfrac{\tan x - \tan y}{1+(\tan x \cdot \tan y)}$.

If we replace x with zero, the formula becomes $\tan(0-y) = \dfrac{\tan 0 - \tan y}{1+\tan 0 \cdot \tan y}$. (3)

Hence, $\tan(-y) = \dfrac{-\tan y}{1}$ and we can write **$\tan(-\alpha) = -\tan\alpha$**.

Result (1), (2) and (3) show how we can write the trigonometric value of the sum or difference of a quadrantal angle and a variable angle (such as $\dfrac{\pi}{2}+\theta$, $\dfrac{3\pi}{2}-y$, $\pi+\alpha$ etc.) as a trigonometric value of the single variable angle. In each case we 'reduced' the trigonometric ratio of a sum or difference to a ratio for a single angle, and for this reason formulas such as (1), (2) and (3) above are called **reduction formulas**.

In a similar manner we can derive all of the reduction formulas by substituting quadrantal angles into the sum and difference formulas.

Reduction Formulas for the First Quadrant

$\sin(0+\alpha) = \sin\alpha$	$\sin\left(\dfrac{\pi}{2}-\alpha\right) = \cos\alpha$
$\cos(0+\alpha) = \cos\alpha$	$\cos\left(\dfrac{\pi}{2}-\alpha\right) = \sin\alpha$
$\tan(0+\alpha) = \tan\alpha$	$\tan\left(\dfrac{\pi}{2}-\alpha\right) = \cot\alpha$
$\cot(0+\alpha) = \cot\alpha$	$\cot\left(\dfrac{\pi}{2}-\alpha\right) = \tan\alpha$

Reduction Formulas for the Second Quadrant

$\sin(\pi-\alpha) = \sin\alpha$	$\sin\left(\dfrac{\pi}{2}+\alpha\right) = \cos\alpha$
$\cos(\pi-\alpha) = -\cos\alpha$	$\cos\left(\dfrac{\pi}{2}+\alpha\right) = -\sin\alpha$
$\tan(\pi-\alpha) = -\tan\alpha$	$\tan\left(\dfrac{\pi}{2}+\alpha\right) = -\cot\alpha$
$\cot(\pi-\alpha) = -\cot\alpha$	$\cot\left(\dfrac{\pi}{2}+\alpha\right) = -\tan\alpha$

Reduction Formulas for the Third Quadrant

$\sin(\pi + \alpha) = -\sin\alpha$	$\sin\left(\frac{3\pi}{2} - \alpha\right) = -\cos\alpha$
$\cos(\pi + \alpha) = -\cos\alpha$	$\cos\left(\frac{3\pi}{2} - \alpha\right) = -\sin\alpha$
$\tan(\pi + \alpha) = \tan\alpha$	$\tan\left(\frac{3\pi}{2} - \alpha\right) = \cot\alpha$
$\cot(\pi + \alpha) = \cot\alpha$	$\cot\left(\frac{3\pi}{2} - \alpha\right) = \tan\alpha$

Reduction Formulas for the Fourth Quadrant

$\sin(2\pi - \alpha)$ or $\sin(0 - \alpha)$ or $\sin(-\alpha) = -\sin\alpha$	$\sin\left(\frac{3\pi}{2} + \alpha\right) = -\cos\alpha$
$\cos(2\pi - \alpha)$ or $\cos(0 - \alpha)$ or $\cos(-\alpha) = \cos\alpha$	$\cos\left(\frac{3\pi}{2} + \alpha\right) = \sin\alpha$
$\tan(2\pi - \alpha)$ or $\tan(0 - \alpha)$ or $\tan(-\alpha) = -\tan\alpha$	$\tan\left(\frac{3\pi}{2} + \alpha\right) = -\cot\alpha$
$\cot(2\pi - \alpha)$ or $\cot(0 - \alpha)$ or $\cot(-\alpha) = -\cot\alpha$	$\cot\left(\frac{3\pi}{2} + \alpha\right) = -\tan\alpha$

Note that reduction formulas are only really useful for problems which ask us to a reduce a trigonometric ratio with a variable angle such as α, θ, x, y etc. If the ratio is simply a numerical angle (e.g. 135°, 180° etc.) it is easier to calculate the value from the reference angle.

Remember!

Sine and cosine are cofunctions. The other cofunction pairs are tangent and cotangent; secant and cosecant.

Of course, it is difficult to remember all of the reduction formulas. Instead, we can derive them from the sum and difference formulas, or we can follow the steps below.

1. Find the primary directed angle for the given angle and determine its quadrant.
2. Determine the sign of the function in the corresponding quadrant.
3. Express the angle or expression as a sum or difference with a suitable quadrantal angle.
4. **a.** Use the sign from step 2.
 b. If the quadrantal angle is an **odd** multiple of $\frac{\pi}{2}$ or 90°, replace the initial function with its **cofunction**. If the quadrantal angle is an **even** multiple of $\frac{\pi}{2}$ or 90°, keep the original function.
 c. Eliminate the quadrantal angle.
5. Find the trigonometric value of the remaining angle.

EXAMPLE 88 Reduce each trigonometric value $(0 < \alpha < \frac{\pi}{2})$.

a. $\sin 585°$ b. $\cot 300°$ c. $\cos\left(\frac{\pi}{2} + \alpha\right)$ d. $\tan(2\pi - \alpha)$

Solution We will use the procedure given above.

a. 1. $585° = 360° + 225$, third quadrant

2. In the third quadrant, the sine function is negative.

3. $\sin 225° = \sin(180° + 45°)$

4. a. sine is negative: $\sin 225° = -$

b. 180° is an even multiple of 90° so use the original function: $\sin 225° = -\sin$

c. take away the quadrantal angle: $\sin 225° = -\sin 45°$

5. $\sin 225° = -\sin 45° = -\frac{\sqrt{2}}{2}$

b. 1. 300° is already a primary directed angle, fourth quadrant

2. In the fourth quadrant, the cotangent function is negative.

3. $\cot 300° = \cot(270° + 30°)$

4. a. $\cot 300° = -$ (cotangent function is negative)

b. $\cot 300° = -\tan$ (270° is an odd multiple of 90°: use the cofunction)

c. $\cot 300° = -\tan 30°$ (take away the quadrantal angle)

5. $\cot 300° = -\tan 30° = -\frac{\sqrt{3}}{3}$

c. 1. primary directed angle, second quadrant

2. In the second quadrant, the cosine function is negative.

3. $\cos\left(\frac{\pi}{2} + \alpha\right)$

4. a. $\cos\left(\frac{\pi}{2} + \alpha\right) = -$ (cosine function is negative)

b. $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin$ (odd multiple of $\frac{\pi}{2}$: use the cofunction)

c. $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha$ (take away the quadrantal angle)

5. $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha$

Note that we used the sign of the original function (cosine) in the corresponding quadrant. We do not need to consider the sign of the cofunction.



d. 1. primary directed angle, fourth quadrant

2. In the fourth quadrant, the tangent function is negative.

3. $\tan(2\pi - \alpha)$

4. a. $\tan(2\pi - \alpha) = -$ (tangent function is negative)

b. $\tan(2\pi - \alpha) = -\tan$ (even multiple of $\frac{\pi}{2}$: use the original function)

c. $\tan(2\pi - \alpha) = -\tan\alpha$ (remove the quadrantal angle)

5. $\tan(2\pi - \alpha) = -\tan\alpha$

EXAMPLE 89 Simplify the expressions.

a. $\cot 240° + \tan 150° + \cos 315° + \sin 750°$

b. $\cos(49\pi - \alpha) + \cot(-100\pi + \alpha) + \sin\left(\frac{71\pi}{2} + \alpha\right) + \tan\left(-\frac{37\pi}{2} - \alpha\right)$

Solution a. We have numerical angle measures. We can use the reference angle or the reduction process to evaluate each term separately.

$$\cot 240° = \cot(180° + 60°) = \cot 60° = \frac{\sqrt{3}}{3}$$

$$\tan 150° = \tan(180° - 30°) = -\tan 30° = -\frac{\sqrt{3}}{3}$$

$$\cos 315° = \cos(270° + 45°) = \sin 45° = \frac{\sqrt{2}}{2}$$

$$\sin 750° = \sin(2 \cdot 360° + 30°) = \sin 30° = \frac{1}{2}$$

Combining these results gives us

$$\cot 240° + \tan 150° + \cos 315° + \sin 750° = \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3} + \frac{\sqrt{2}}{2} + \frac{1}{2}$$

$$= \frac{\sqrt{2}+1}{2}.$$

b. We have variable angle measures so we need to use the reduction process on each term.

$\cos(49\pi - \alpha) = \cos[(24 \cdot 2\pi) + \pi - \alpha] = \cos(\pi - \alpha) = -\cos\alpha$

$\cot(-100\pi + \alpha) = \cot[(-50 \cdot 2\pi) + \alpha] = \cot(0 + \alpha) = \cot\alpha$

$$\sin\left(\frac{71\pi}{2} + \alpha\right) = \sin\left((17 \cdot 2\pi) + \frac{3\pi}{2} + \alpha\right) = \sin\left(\frac{3\pi}{2} + \alpha\right) = -\cos\alpha$$

$$\tan\left(-\frac{37\pi}{2} - \alpha\right) = \tan\left((-10 \cdot 2\pi) + \frac{3\pi}{2} - \alpha\right) = \tan\left(\frac{3\pi}{2} - \alpha\right) = \cot\alpha$$

In conclusion, $-\cos\alpha + \cot\alpha - \cos\alpha + \cot\alpha = 2(\cot\alpha - \cos\alpha)$.

Check Yourself 21

Simplify the expressions.

1. $\tan 1200° + \cot 2010° + \sin(-390°) + \cos(-780°)$
2. $\sin\left(\frac{\pi}{2} - \alpha\right) + \cos(\pi - \alpha) + \tan\left(\frac{3\pi}{2} + \alpha\right) + \cot(-\alpha)$
3. $\sin\left(-\frac{\pi}{2} - \alpha\right) + \cos\left(-\frac{\pi}{2} + \alpha\right)$
4. $\sin 150° + \cos 240° + \tan 210° + \cot 300°$

Answers

1. 0 2. $-2\cot\alpha$ 3. $-\cos\alpha + \sin\alpha$ 4. 0

4. Sum to Product Formulas

In this section we will study formulas which allow us to rewrite the sum of two trigonometric functions as a product of two trigonometric functions. We will use the sum and difference formulas in our working.

a. sin a ± sin b

To obtain a formula for $\sin a + \sin b$ we add the formulas for the sine of a sum and the sine of a difference:

$$\sin(x+y) = (\sin x \cdot \cos y) + (\cos x \cdot \sin y)$$

$$+\sin(x-y) = +(\sin x \cdot \cos y) - (\cos x \cdot \sin y)$$

$$\sin(x+y) + \sin(x-y) = 2 \cdot \sin x \cdot \cos y. \qquad (1)$$

If we write $a = x + y$ and $b = x - y$ we obtain the system $\begin{cases} a = x + y \\ b = x - y \end{cases}$.

Solving this system for x and y gives $x = \frac{a+b}{2}$ and $y = \frac{a-b}{2}$, which we can substitute in (1):

$$\sin a + \sin b = 2 \sin\frac{a+b}{2} \cos\frac{a-b}{2}. \qquad (2)$$

If we subtract the two formulas instead of adding them, we get

$$\sin(x+y) = (\sin x \cdot \cos y) + (\cos x \cdot \sin y)$$

$$-\sin(x-y) = -(\sin x \cdot \cos y) - (\cos x \cdot \sin y)$$

$$\sin(x+y) - \sin(x-y) = 2 \cdot \cos x \cdot \sin y. \qquad (3)$$

If we write $a = x + y$ and $b = x - y$ we obtain the system

$\begin{cases} a = x+y \\ b = x-y \end{cases}$. Solving this gives $x = \dfrac{a+b}{2}$ and $y = \dfrac{a-b}{2}$, which we can substitute in (3) :

$$\sin a - \sin b = 2 \cos \frac{a+b}{2} \sin \frac{a-b}{2}. \qquad (4)$$

EXAMPLE 20 Evaluate the expressions.

a. $\sin 105° + \sin 15°$

b. $\sin 70° - \sin 10°$

Solution We could evaluate each term separately and then combine them. Howewer, the sum to product formulas give us an easier way.

a. $$\sin 105° + \sin 15° = 2 \cdot \sin\left(\frac{105°+15°}{2}\right) \cdot \cos\left(\frac{105°-15°}{2}\right)$$

$$= 2 \cdot \sin 60° \cdot \cos 45°$$

$$= 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6}}{2}$$

b. $$\sin 70° - \sin 10° = 2 \cdot \cos\left(\frac{70°-10}{2}\right) \cdot \sin\left(\frac{70°+10°}{2}\right)$$

$$= 2 \cdot \cos 30° \cdot \sin 40°$$

$$= 2 \cdot \frac{\sqrt{3}}{2} \cdot \sin 40°$$

$$= \sqrt{3} \cdot \sin 40°$$

b. cos a ± cos b

Now let us add the formulas for the cosine of a sum and the cosine of a difference:

$$\begin{aligned} \cos(x+y) &= (\cos x \cdot \cos y) - (\sin x \cdot \sin y) \\ \underline{+\cos(x-y)} &= \underline{+(\cos x \cdot \cos y) + (\sin x \cdot \sin y)} \end{aligned}$$

$$\cos(x+y) + \cos(x-y) = 2 \cdot \cos x \cdot \cos y. \qquad (5)$$

If we write $a = x + y$ and $b = x - y$ then we obtain the system

$\begin{cases} a = x+y \\ b = x-y \end{cases}$. Solving this gives $x = \dfrac{a+b}{2}$ and $y = \dfrac{a-b}{2}$, which we can substitute in (5):

$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}. \qquad (6)$$

If we subtract the formulas instead of adding them, we get

$$\begin{aligned} \cos(x+y) &= (\cos x \cdot \cos y) - (\sin x \cdot \sin y) \\ \underline{-\cos(x-y)} &= \underline{-(\cos x \cdot \cos y) + (\sin x \cdot \sin y)} \end{aligned}$$

$$\cos(x+y) - \cos(x-y) = -2 \cdot \sin x \cdot \sin y. \qquad (7)$$

If we write $a = x + y$ and $b = x - y$ we obtain the system

$\begin{cases} a = x+y \\ b = x-y \end{cases}$. Solving gives $x = \dfrac{a+b}{2}$ and $y = \dfrac{a-b}{2}$ which we can substitute in (7):

$$\cos a - \cos b = -2\sin\frac{a+b}{2}\sin\frac{a-b}{2}. \qquad (8)$$

EXAMPLE 91 Simplify $\dfrac{\cos 105° + \cos 15°}{\sin 105° + \sin 15°}$.

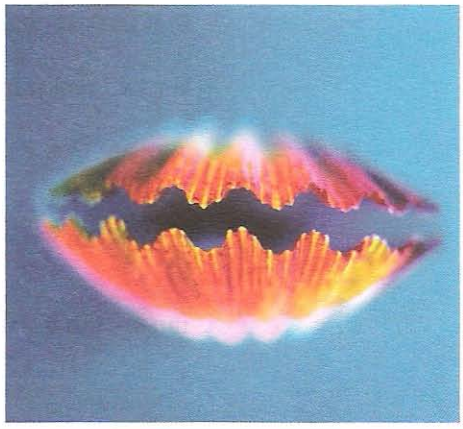
Solution

$$\frac{\cos 105° + \cos 15°}{\sin 105° + \sin 15°} = \frac{2\cos\frac{105°+15°}{2} \cdot \cos\frac{105°-15°}{2}}{2\sin\frac{105°+15°}{2} \cdot \cos\frac{105°-15°}{2}} \quad \text{(sum to product formulas)}$$

$$= \frac{\cos 60° \cdot \cos 45°}{\sin 60° \cdot \cos 45°}$$

$$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}.$$

EXAMPLE 92 Evaluate $\dfrac{\sin a - \sin b}{\cos a - \cos b}$ given that $a + b = \dfrac{\pi}{3}$.



Solution

$$\frac{\sin a - \sin b}{\cos a - \cos b} = \frac{2\cos\dfrac{a+b}{2}\sin\dfrac{a-b}{2}}{-2\sin\dfrac{a+b}{2}\sin\dfrac{a-b}{2}}$$

$$= \frac{\cos\dfrac{a+b}{2}}{-\sin\dfrac{a+b}{2}} = -\cot\frac{a+b}{2}$$

$$= -\cot\frac{a+b}{2} = -\cot\frac{\dfrac{\pi}{3}}{2}$$

$$= -\cot\frac{\pi}{6} = -\sqrt{3}$$

EXAMPLE 93 Find the simplest form of the expression $\dfrac{\sin 3x + \sin 5x + \sin 7x}{\cos 3x + \cos 5x + \cos 7x}$.

Solution We can use the first and last terms of both the numerator and the denominator in the sum to product formulas.

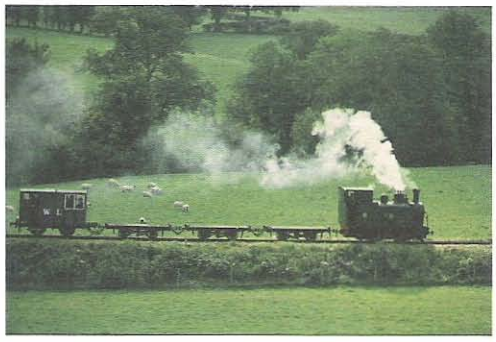
$$\frac{\sin 3x + \sin 5x + \sin 7x}{\cos 3x + \cos 5x + \cos 7x} = \frac{2\sin\dfrac{7x+3x}{2}\cos\dfrac{7x-3x}{2} + \sin 5x}{2\cos\dfrac{7x+3x}{2}\cos\dfrac{7x-3x}{2} + \cos 5x}$$

$$= \frac{2\sin 5x \cdot \cos 2x + \sin 5x}{2\cos 5x \cdot \cos 2x + \cos 5x}$$

$$= \frac{\sin 5x \cdot (2\cos 2x + 1)}{\cos 5x \cdot (2\cos 2x + 1)}$$

$$= \frac{\sin 5x}{\cos 5x} = \tan 5x$$

c. tan a ± tan b



For two real numbers a and b not equal to

$k\pi + \frac{\pi}{2}$ $(k \in \mathbb{Z})$, we know that

$\tan a + \tan b = \frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}$. Multiplying gives us

$\tan a + \tan b = \frac{(\sin a \cdot \cos b) + (\cos a \cdot \sin b)}{\cos a \cdot \cos b}$, i.e.

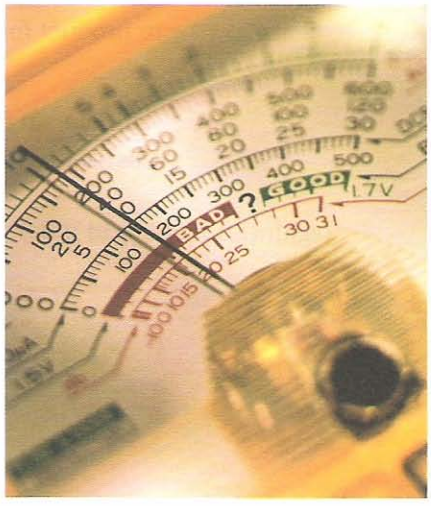
$$\tan a + \tan b = \frac{\sin (a+b)}{\cos a \cdot \cos b}. \qquad (9)$$

If we write $-b$ instead of b in this equation, we get

$\tan a + \tan (-b) = \frac{\sin (a + (-b))}{\cos a \cdot \cos (-b)}$, i.e.

$$\tan a - \tan b = \frac{\sin (a-b)}{\cos a \cdot \cos b}. \qquad (10)$$

EXAMPLE 94 Evaluate tan 75° + tan 15°.



Solution We can use the sum to product formula for the tangent.

$$\tan 75° + \tan 15° = \frac{\sin (75° + 15°)}{\cos 75° \cdot \cos 15°}$$

$$= \frac{\sin (90°)}{\sin 15° \cdot \cos 15°}$$

$$= \frac{1}{\frac{\sin 30°}{2}}$$

$$= \frac{1}{\frac{1}{2} \cdot \frac{1}{2}} = 4$$

EXAMPLE 95 Given $\sin 10° = t$, evaluate $\tan 70° + \tan 10°$ in terms of t.

Solution

$$\tan 70° + \tan 10° = \frac{\sin(70° + 10°)}{\cos 70° \cdot \cos 10°}$$

$$= \frac{\sin(80°)}{\cos 70° \cdot \cos 10°}$$

$$= \frac{\cos 10°}{\sin 20° \cdot \cos 10°} = \frac{1}{\sin 20°}$$

$$= \frac{1}{2\sin 10° \cdot \cos 10°} = \frac{1}{2 \cdot t \cdot \sqrt{1 - t^2}}$$

d. cot a ± cot b

For two real numbers a and b not equal to $k\pi + \frac{\pi}{2}$ $(k \in \mathbb{Z})$, we know that

$$\cot a + \cot b = \frac{\cos a}{\sin a} + \frac{\cos b}{\sin b}.$$

Multiplying gives us $\cot a + \cot b = \frac{\cos a \cdot \sin b + \cos b \cdot \sin a}{\sin a \cdot \sin b}$, i.e.

$$\cot a + \cot b = \frac{\sin(a + b)}{\sin a \cdot \sin b}. \qquad (11)$$

If we write $-b$ instead of b in this equation, we get

$$\cot a + \cot(-b) = \frac{\sin(a + (-b))}{\sin a \cdot \sin(-b)}, \text{ i.e.}$$

$$\cot a - \cot b = -\frac{\sin(a - b)}{\sin a \cdot \sin b}. \qquad (12)$$

EXAMPLE 96 $18x = \pi$ is given. Find the simplest form of the expression $\frac{\cot 9x + \cot x}{\cot 9x - \cot x}$.

Solution

$$\frac{\cot 9x + \cot x}{\cot 9x - \cot x} = \frac{\frac{\sin(9x + x)}{\sin 9x \cdot \sin x}}{-\frac{\sin(9x - x)}{\sin 9x \cdot \sin x}} = -\frac{\sin 10x}{\sin 8x}$$

$$= -\frac{\sin(18x - 8x)}{\sin 8x}$$

$$= -\frac{\sin(\pi - 8x)}{\sin 8x}$$

$$= -\frac{\sin 8x}{\sin 8x} = -1$$

Check Yourself 22

Simplify each expression.

1. $\sin 75° - \sin 15°$
2. $\cos 105° - \cos 75°$
3. $\tan 165° + \tan 15°$
4. $\cot 75° - \cot 195°$

Answers

1. $\frac{\sqrt{2}}{2}$ 2. $-2\sin 15° = \frac{\sqrt{2}-\sqrt{6}}{2}$ 3. 0 4. $-2\sqrt{3}$



5. Product to Sum Formulas

If we need to convert a product of two trigonometric expressions to a sum or difference we use the product to sum formulas. These formulas are derived from the sum and difference formulas.

a. $\sin x \cdot \sin y$

For any real numbers x and y,

$$\cos(x+y) = (\cos x \cdot \cos y) - (\sin x \cdot \sin y)$$

$$-\cos(x-y) = -(\cos x \cdot \cos y) + (\sin x \cdot \sin y)$$

$$\cos(x+y) - \cos(x-y) = -2 \cdot \sin x \cdot \sin y$$

$$\sin x \cdot \sin y = -\frac{1}{2}[\cos(x+y) - \cos(x-y)]. \qquad (13)$$

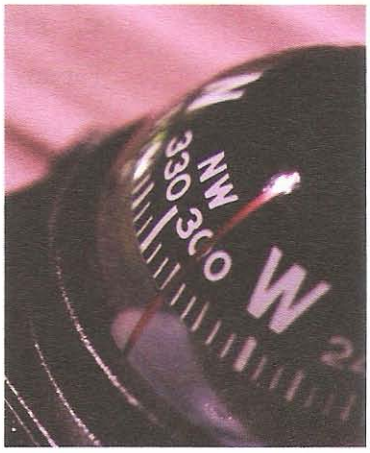
EXAMPLE 97 Write $\sin 5\alpha \cdot \sin 3\alpha$ as the difference of two trigonometric ratios.

Solution Let us use the product to sum formula for sine:

$$\sin 5\alpha \cdot \sin 3\alpha = -\frac{1}{2}[\cos(5\alpha + 3\alpha) - \cos(5\alpha - 3\alpha)]$$

$$= -\frac{1}{2}[\cos(8\alpha) - \cos(2\alpha)].$$

EXAMPLE 98 Evaluate $\sin 75° \cdot \sin 15°$.



Solution We could evaluate the two terms separately. However, let us use the product to sum formula for sine:

$$\sin 75° \cdot \sin 15° = -\frac{1}{2}[\cos(75° + 15°) - \cos(75° - 15°)]$$

$$= -\frac{1}{2}[\cos 90° - \cos 60°]$$

$$= -\frac{1}{2}\left[0 - \frac{1}{2}\right] = \frac{1}{4}.$$

b. cos x · cos y

For any real numbers x and y,

$$\cos(x+y) = (\cos x \cdot \cos y) - (\sin x \cdot \sin y)$$

$$+\cos(x-y) = +(\cos x \cdot \cos y) + (\sin x \cdot \sin y)$$

$$\cos(x+y) + \cos(x-y) = 2 \cdot \cos x \cdot \cos y$$

$$\cos x \cdot \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]. \qquad (14)$$

EXAMPLE 99 Write $\cos 7\alpha \cdot \cos 4\alpha$ as the sum of two trigonometric ratios.

Solution By the product to sum formula for cosine,

$$\cos 7\alpha \cdot \cos 4\alpha = \frac{1}{2}[\cos(7\alpha + 4\alpha) + \cos(7\alpha - 4\alpha)]$$

$$= \frac{1}{2}[\cos(11\alpha) + \cos(3\alpha)].$$

EXAMPLE 100 Evaluate $\cos 105° \cdot \cos 15°$.

Solution $$\cos 105° \cdot \cos 15° = \frac{1}{2}[\cos(105° + 15°) + \cos(105° - 15°)]$$

$$= \frac{1}{2}[\cos 120° + \cos 90°]$$

$$= \frac{1}{2}\left[-\frac{1}{2} + 0\right] = -\frac{1}{4}$$

c. sin x · cos y

For any real numbers x and y,

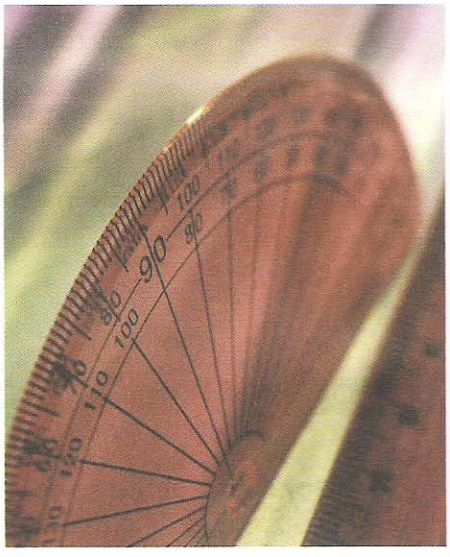
$$\sin(x+y) = (\sin x \cdot \cos y) + (\cos x \cdot \sin y)$$

$$+\sin(x-y) = +(\sin x \cdot \cos y) - (\cos x \cdot \sin y)$$

$$\sin(x+y) + \sin(x-y) = 2 \cdot \sin x \cdot \cos y$$

$$\sin x \cdot \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)] . \qquad (15)$$

EXAMPLE 101 Evaluate $\sin\frac{3\pi}{8} \cdot \cos\frac{\pi}{8}$.



Solution By the product to sum formula,

$$\sin\frac{3\pi}{8} \cdot \cos\frac{\pi}{8} = \frac{1}{2}\left[\sin\left(\frac{3\pi}{8}+\frac{\pi}{8}\right) + \sin\left(\frac{3\pi}{8}-\frac{\pi}{8}\right)\right]$$

$$= \frac{1}{2}\left[\sin\frac{\pi}{2} + \sin\frac{\pi}{4}\right]$$

$$= \frac{1}{2}\left[1+\frac{\sqrt{2}}{2}\right] = \frac{2+\sqrt{2}}{4}.$$

d. cos x · sin y

For any real numbers x and y,

$$\sin(x+y) = (\sin x \cdot \cos y) + (\cos x \cdot \sin y)$$

$$-\sin(x-y) = -(\sin x \cdot \cos y) - (\cos x \cdot \sin y)$$

$$\sin(x+y) - \sin(x-y) = 2 \cdot \cos x \cdot \sin y$$

$$\cos x \cdot \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)] . \qquad (16)$$

EXAMPLE 102 Evaluate $\cos\frac{\pi}{8}\cdot\sin\frac{3\pi}{8}$.

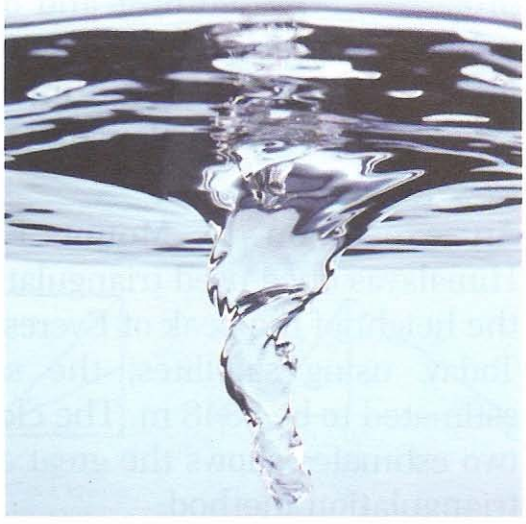
Solution
$$\cos\frac{\pi}{8}\cdot\sin\frac{3\pi}{8}=\frac{1}{2}\left[\sin\left(\frac{\pi}{8}+\frac{3\pi}{8}\right)-\sin\left(\frac{\pi}{8}-\frac{3\pi}{8}\right)\right]$$
$$=\frac{1}{2}\left[\sin\frac{\pi}{2}-\sin\left(-\frac{\pi}{4}\right)\right]=\frac{1}{2}\left[\sin\frac{\pi}{2}+\sin\frac{\pi}{4}\right]$$
$$=\frac{1}{2}\left[1+\frac{\sqrt{2}}{2}\right]=\frac{2+\sqrt{2}}{4}$$

Note

Example 101 and Example 102 are actually the same question. The cosine and sine functions were simply in a different order, so the result does not change due to the commutative property of multiplication. Although we applied a different formula in each question, the results are the same. Therefore we can say that formulas (15) and (16) are different forms of the same identity.

EXAMPLE 103 Evaluate $\cos 15°\cdot\sin 75°$.

Solution
$$\cos 15°\cdot\sin 75°=\frac{1}{2}\left[\sin(15°+75°)-\sin(15°-75°)\right]$$
$$=\frac{1}{2}\left[\sin 90°-\sin(-60°)\right]=\frac{1}{2}\left[\sin 90°+\sin 60°\right]\qquad(\sin(-60°)=-\sin 60°)$$
$$=\frac{1}{2}\left[1+\frac{\sqrt{3}}{2}\right]=\frac{2+\sqrt{3}}{4}$$



Check Yourself 23

Evaluate each expression.

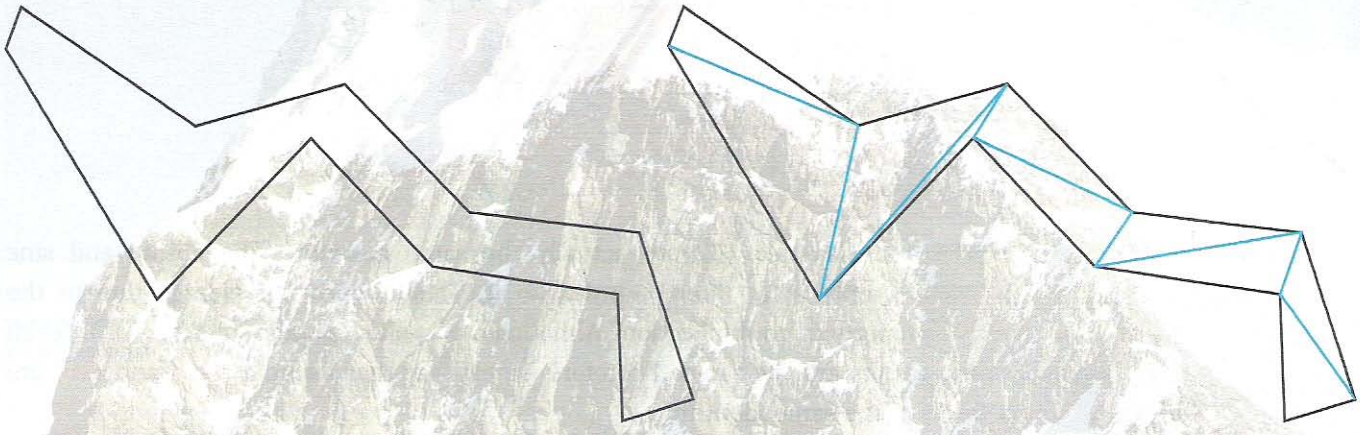
1. $\sin 105°\cdot\sin 15°$
2. $\cos 75°\cdot\cos 15°$
3. $\sin 75°\cdot\cos 15°$

Answers

1. $\frac{1}{4}$ 2. $\frac{1}{4}$ 3. $\frac{2+\sqrt{3}}{4}$

TRIANGULATION AND SURVEYING

Triangulation is the process of dividing a polygon in a plane into a set of triangles, usually with the restriction that each triangle side is shared completely by two adjacent triangles.



The process is started by measuring the length of an initial baseline between two surveying stations. Then, using an instrument called a **theodolite**, a surveyor measures the angles between these two stations and a third station. The law of sines is then used to calculate the two other sides of the triangle formed by the three stations. The calculated sides are used as baselines, and the process is repeated over and over to create a network of triangles. In this method, the only distance measured is the initial baseline. All other distances are calculated using the law of sines.

An expedition to Mount Everest in the Himalayas once used triangulation to calculate the height of the peak of Everest to be 8840 m. Today, using satellites, the same height is estimated to be 8848 m. The closeness of these two estimates shows the great accuracy of the triangulation method.



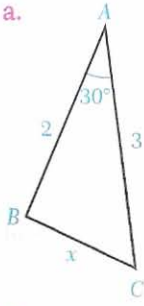
a theodolite

EXERCISES 4

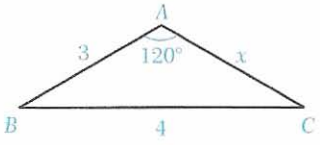
A. Trigonometric Theorems

1. Find the length of side x in each triangle.

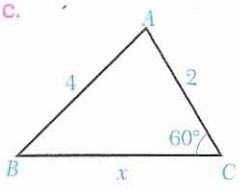
a.



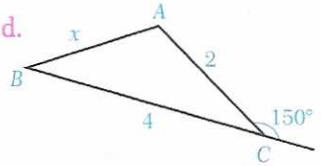
b.



c.

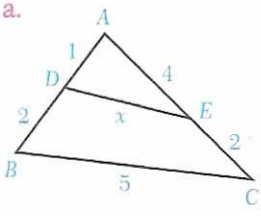


d.

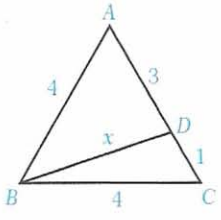


2. Find the length x in each figure.

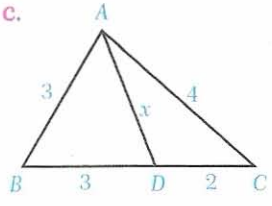
a.



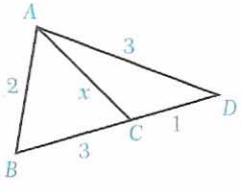
b.



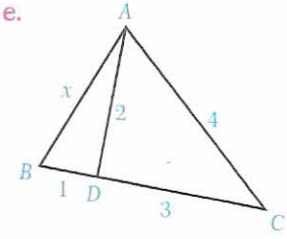
c.



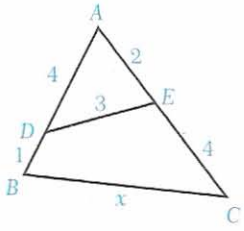
d.



e.



f.

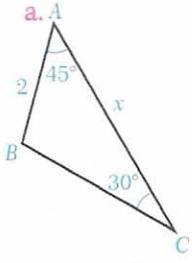


3. A triangle ABC has sides a, b and c such that $a^2 - b^2 - c^2 - (b \cdot c) = 0$. Find $m(\angle A) = \alpha$.

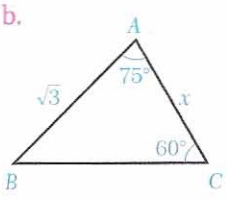
4. A triangle ABC has sides a, b and c such that $c^2 - b^2 = a^2 + \sqrt{2} \cdot b \cdot a$. Find $m(\angle C) = \alpha$.

5. Find the length x in each figure.

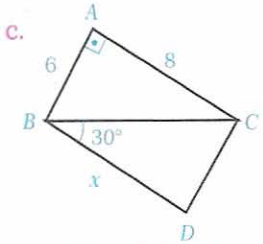
a.



b.



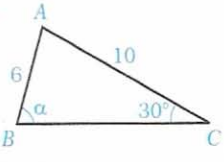
c.



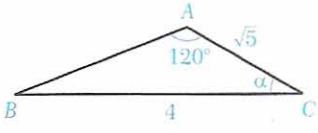
A(ΔABC) = A(ΔDBC)

6. Calculate sin α in each figure.

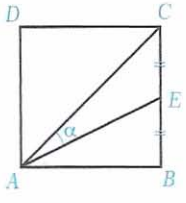
a.



b.



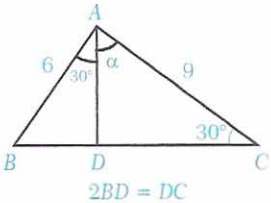
c.



ABCD is a square

$CE = BE$

d.

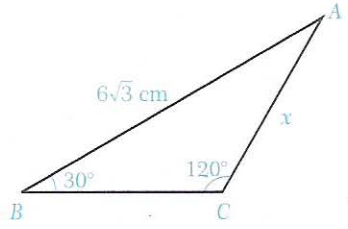


$2BD = DC$

7. The side lengths of a triangle ABC are 3, 7 and 8 units respectively. What is the circumference of the circumscribed circle of the triangle?

8. In a triangle ABC, $m(\angle A) = 75°$, $m(\angle B) = 60°$ and $AB = c = 10$ cm.
 a. Find the length of the side b.
 b. Find the radius of the circumscribed circle of the triangle.

9. In a triangle ABC, $m(\angle ACB) = 120°$, $m(\angle ABC) = 30°$ and $AB = 6\sqrt{3}$ cm. Find AC.



10. A triangle ABC has side lengths $a = 3$, $b = 3\sqrt{3}$ and $c = 6$. Its circumscribed circle has radius $R = 3$. Find the measure of each interior angle of the triangle.

11. A triangle ABC has sides $a = 7$ cm, $b = 24$ cm and $c = 25$ cm. Find $A(\Delta ABC)$.

12. A triangle ABC has sides of length 8, 15 and 17 units respectively. Find the radius of its circumscribed circle.

B. Trigonometric Formulas

13. Calculate the values without using a trigonometric table or a calculator.
 a. $\sin 105°$ b. $\cos 15°$ c. $\tan 75°$
 d. $\cos 105°$ e. $\tan 165°$ f. $\cot 255°$
 g. $\sin 195°$ h. $\cot 345°$

14. $\sin x = \frac{2}{3}$ and $\cos y = \frac{1}{4}$ are given. Find the value of each expression.
 a. $\sin (x + y)$ b. $\sin (x - y)$
 c. $\cos (x + y)$ d. $\cos (x - y)$
 e. $\tan (x + y)$ f. $\cot (x - y)$

15. ✪ x and y are acute angles such that $\tan x = \frac{1}{4}$ and $\tan y = \frac{3}{5}$. Evaluate the expressions.
 a. $\tan (x + y)$ b. $\cos (2x + y)$
 c. $\sin (x + y)$ d. $\cos (x - 2y)$
 e. $\cot (2x + 2y)$ f. $\sin (x + 2y)$

16. ✪ Simplify the expressions.
 a. $\sin (x + 30°) + \cos (x + 60°)$
 b. $\cos (x + y) + \cos (x - y)$
 c. $\sin (x + 30°) + \sin (x - 30°)$
 d. $\sin (x + y) - \sin (x - y)$

17. ✪ Express $\cos 3\alpha$ in terms of $\cos \alpha$ and $\tan 3\alpha$ in terms of $\tan \alpha$.

18. Calculate $\sin 2x$, $\cos 2x$ and $\tan 2x$ from the information given in each question.

a. $\sin x = \frac{3}{5}$ and $x \in \left(0, \frac{\pi}{2}\right)$

b. $\cos x = \frac{5}{13}$ and $\csc x < 0$

c. $\tan x = -\frac{7}{24}$ and $x \in \left(\frac{\pi}{2}, \pi\right)$

d. $\csc x = 4$ and $\tan x < 0$

e. $\cot x = \frac{2}{3}$ and $\sin x > 0$

19. Calculate $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ from the information given in each question.

a. $\sin x = \frac{4}{5}$ and $x \in \left(0, \frac{\pi}{2}\right)$

b. $\cos x = -\frac{3}{5}$ and $x \in \left(\pi, \frac{3\pi}{2}\right)$

c. $\csc x = 3$ and $x \in \left(\frac{\pi}{2}, \pi\right)$

d. $\tan x = 1$ and $x \in \left(0, \frac{\pi}{2}\right)$

e. $\sec x = \frac{3}{2}$ and $x \in \left(\frac{3\pi}{2}, 2\pi\right)$

f. $\cot x = 5$ and $\csc x < 0$

20. Simplify each expression.

a. $\sin 105° - \sin 15°$

b. $\cos 75° + \cos 15°$

c. $\cos 105° - \cos 15°$

d. $\sin 165° + \sin 15°$

e. $\sin 75° + \sin 195°$

f. $\sin 105° + \sin 255°$

g. $\cos \frac{\pi}{12} - \cos \frac{5\pi}{12}$

h. $\sin \frac{3\pi}{8} - \sin \frac{\pi}{8}$

21. Express each sum or difference as a product of trigonometric functions.

a. $\sin 5x + \sin 3x$

b. $\sin x - \sin 4x$

c. $\cos 4x - \cos 6x$

d. $\cos 9x + \cos 2x$

e. $\sin 2x - \sin 7x$

f. $\sin 3x + \sin 4x$

g. $\sin 11x + \sin 9x$

h. $\cos \frac{x}{2} - \cos \frac{5x}{2}$

22. Verify each identity.

(Hint: $1 + \cos x = \cos 0° + \cos x$)

a. $1 + \cos x = 2\cos^2 \frac{x}{2}$

b. $1 - \cos x = 2\sin^2 \frac{x}{2}$

c. $1 + \sin x = 2\sin^2\left(\frac{\pi}{4} + \frac{x}{2}\right)$

d. $1 - \sin x = 2\cos^2\left(\frac{\pi}{4} + \frac{x}{2}\right)$

e. $1 + \tan x = \dfrac{\sqrt{2}\sin\left(\frac{\pi}{4} + x\right)}{\cos x}$

f. $1 - \tan x = \dfrac{\sqrt{2}\sin\left(\frac{\pi}{4} - x\right)}{\cos x}$

g. $\dfrac{1 - \tan x}{1 + \tan x} = \tan\left(\frac{\pi}{4} + x\right)$

h. $1 + \cot x = \dfrac{\sqrt{2}\sin\left(\frac{\pi}{4} + x\right)}{\sin x}$

i. $1 + \cot x = \dfrac{\sqrt{2}\sin\left(\frac{\pi}{4} - x\right)}{\sin x}$

23. Prove each statement.

a. $\dfrac{\sin x+\sin 5x}{\cos x+\cos 5x}=\tan 3x$

b. $\dfrac{\sin 3x+\sin 7x}{\cos 3x-\cos 7x}=\cot 2x$

c. $\dfrac{\sin 10x}{\sin 9x+\sin x}=\dfrac{\cos 5x}{\cos 4x}$

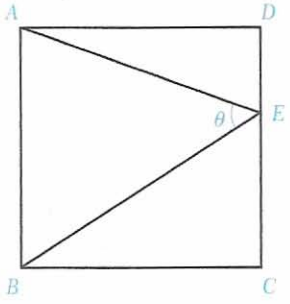
d. $\dfrac{\sin x+\sin 3x+\sin 5x}{\cos x+\cos 3x+\cos 5x}=\tan 3x$

e. $\dfrac{\sin x+\sin y}{\cos x+\cos y}=\tan\dfrac{x+y}{2}$

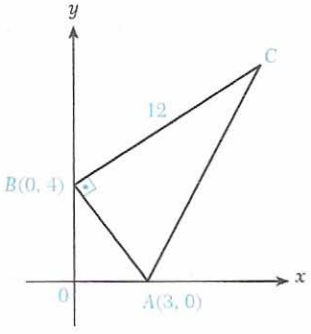
f. $\dfrac{\sin(x+y)-\sin(x-y)}{\cos(x+y)+\cos(x-y)}=\tan y$

g. $\dfrac{\sin(x+y)-\sin(x-y)}{\cos(x-y)+\cos(x+y)}=\cot x$

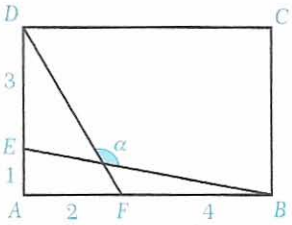
24. In the figure, $ABCD$ is a square with $3DE = 2EC$ and $m(\angle AEB) = \theta$. Find $\tan\theta$.



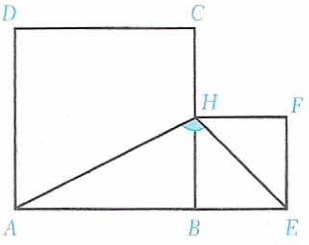
25. In the figure, $CB \perp BA$ and $BC = 12$. Find $\cos(\angle OAC)$.



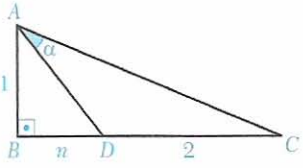
26. In the figure, $ABCD$ is a rectangle. Find $\cot\alpha$.



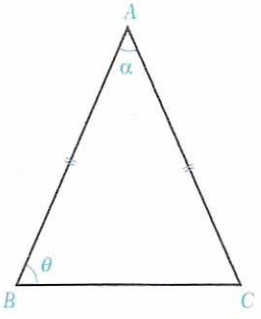
27. In the figure, $ABCD$ and $BEFH$ are squares and $AB = 2BE$. Find $\sin(\angle AHE)$.



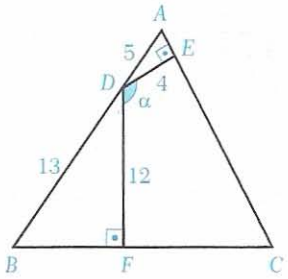
28. Find $\tan\alpha$ in the figure.



29. In the figure opposite, ABC is an isosceles triangle. $AB = AC$, $m(\angle ABC) = \theta$, $m(\angle BAC) = \alpha$ and $\tan\theta = 3$. Find $\tan\alpha$.



30. In the figure,
$AD = 5$ cm,
$BD = 13$ cm,
$DF = 12$ cm,
$DE = 4$ cm and
$m(\angle FDE) = \alpha$. Find
$\cot \alpha$.



31. $\sin 20° = x$ is given. Write $\sin 40°$ in terms of x.

32. ✪ $\tan x - \cot x = \frac{3}{4}$ is given with $0 < x < \frac{\pi}{2}$.
Find $\tan 2x - \cot 2x$.

33. $\cos 2\alpha = -0.125$ is given with $\pi < \alpha < \frac{3\pi}{2}$. Find
$\sin \alpha$.

34. Evaluate $3\cos^2 105° + \sin^2 15°$.

35. $\tan 2\alpha = \frac{4}{3}$ is given and $\frac{\pi}{2} < \alpha < \pi$.
Evaluate $(\sin \alpha \cdot \cos \alpha) - \cos^2 \alpha$.

36. Evaluate $\frac{1}{\sin 15°} + \frac{1}{\cos 15°}$.

37. ✪✪ Evaluate $\sin 10° \cdot \sin 30° \cdot \sin 50° \cdot \sin 70°$.

38. $\cos 76° = x$ is given. Find $\sin 37° \cdot \cos 23°$ in terms of x.

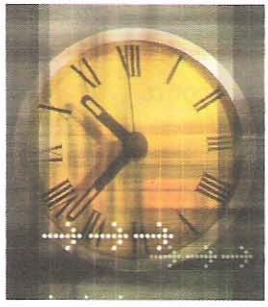
39. $x = \frac{\pi}{12}$ is given. Evaluate $\cos 8x \cdot \cos 2x$.

40. $\pi = 8\alpha$ is given. Evaluate $\cos 5\alpha \cdot \sin 3\alpha$.

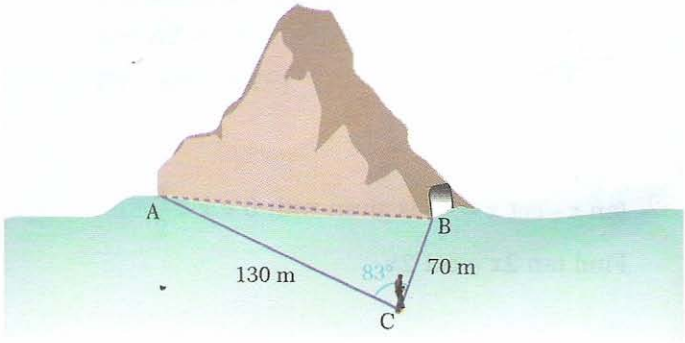
41. ✪ $a - b = \frac{\pi}{6}$ is given. Evaluate
$\tan (2a - b) \cdot \tan (a - 2b)$.

5 APPLIED TRIGONOMETRY

In this section, we will look at some practical applications of trigonometry. In daily life we use many different angles. Therefore we will use a calculator to obtain the trigonometric ratios of the angles in the problems which follow.



EXAMPLE 104 A tunnel is to be built through a mountain. To estimate the length of the tunnel, a surveyor makes the measurements shown in the figure. Use the surveyor's data to approximate the length of the tunnel.



Solution Let c be the length of the tunnel and let us use the law of cosines.

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$ (law of cosines)

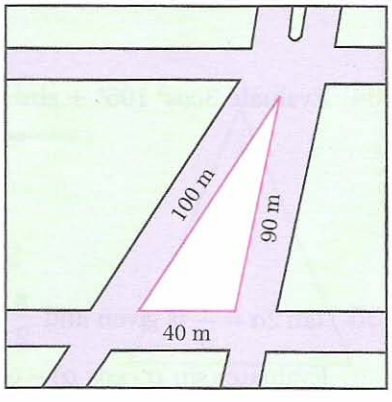
$c^2 = 130^2 + 70^2 - [2 \cdot 130 \cdot 70 \cdot \cos 83°]$ (given)

$c^2 = 19582$ (use a calculator)

$c \cong 139.94$ (take the positive square root)

So the tunnel will be approximately 139.94 m long.

EXAMPLE 105 A businessman wishes to buy a triangular lot in a city as shown in the figure. The lengths of the lot on the three adjacent streets are 40 m, 90 m and 100 m. Find the area of the lot.



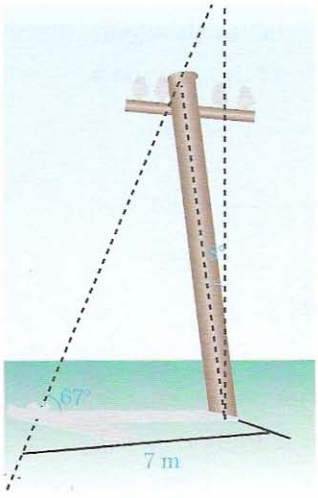
Solution Heron's Formula tells us the area of a triangle from a given perimeter.

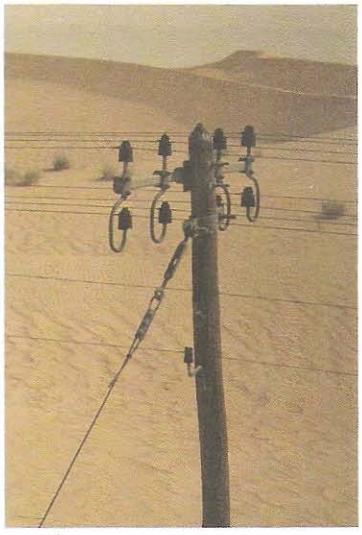
The semiperimeter of the lot is $u = \frac{40+90+100}{2} = 115$, so by Heron's Formula the area is

$$A = \sqrt{115 \cdot (115-100) \cdot (115-90) \cdot (115-40)}$$
$$= \sqrt{115 \cdot 15 \cdot 25 \cdot 75}$$
$$\cong 1798.44.$$

So the area is approximately 1798.44 m^2.

EXAMPLE 106 When the angle of elevation of the sun is 67°, a telephone pole which is tilted at an angle of 8° directly away from the sun has a shadow which is 7 m long on level ground. Find the approximate length of the pole.





Angles of elevation and depression are angles that are formed with the horizontal. If the line of sight is upward from the horizontal, the angle is an angle of elevation. If the line of sight is downward from the horizontal, the angle is an angle of depression.

Solution Triangle ABC in the figure illustrates the problem. Note that in the figure we have calculated the following angles:

$m(\angle ABC) = 90° - 8° = 82°$

$m(\angle ACB) = 180° - (67° + 82°) = 31°$.

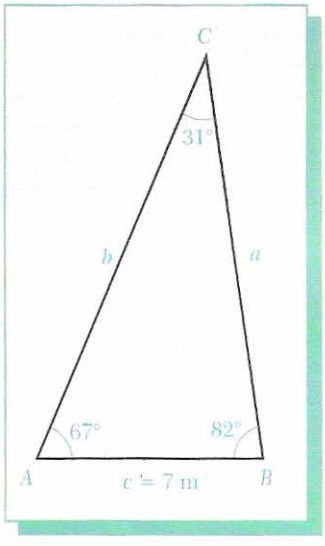
We are asked to find the length of side a in the figure. We can apply the law of sines as follows:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

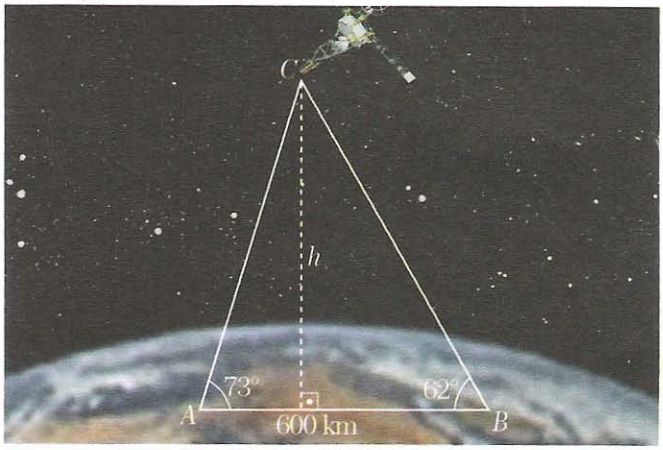
$$\frac{a}{\sin 67°} = \frac{7}{\sin 31°}$$

$$a = \frac{7 \cdot \sin 67°}{\sin 31°} \cong 12.51.$$

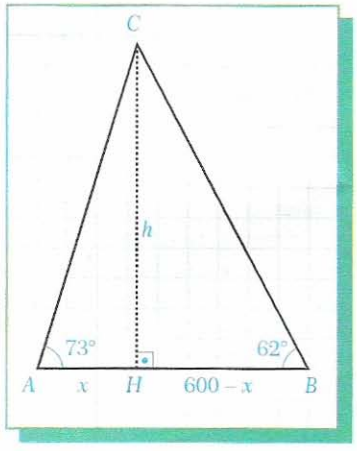
So the telephone pole is approximately 12.51 m long.



EXAMPLE 107 A satellite orbiting the Earth passes directly overhead at observation stations in city A and city B which are 600 km apart. At an instant when the satellite is between these two stations, its angle of elevation is simultaneously observed to be 73° at city A and 62° at city B. Find the altitude of the satellite from the Earth.



Solution Triangle ABC in the figure illustrates the problem. We are asked to find the altitude h.



$$\tan 73° = \frac{h}{x}, \text{ i. e. } h = x \cdot \tan 73° \quad (1)$$

$$\tan 62° = \frac{h}{600 - x}, \text{ i.e.}$$

$$h = (600 - x) \cdot \tan 62° \quad (2)$$

From (1) and (2),

$$x \cdot \tan 73° = (600 - x) \cdot \tan 62°$$

$$x \cdot (3.27) \cong (600 - x) \cdot (1.88)$$

$$5.15x \cong 1128$$

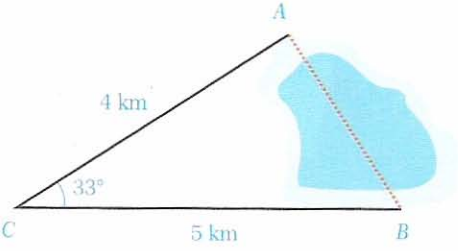
$$x \cong 219.03 \text{ km}.$$

From (1), $h = x \cdot \tan 73°$, so $h \cong 219.03 \cdot (3.27) \cong 716.23$.

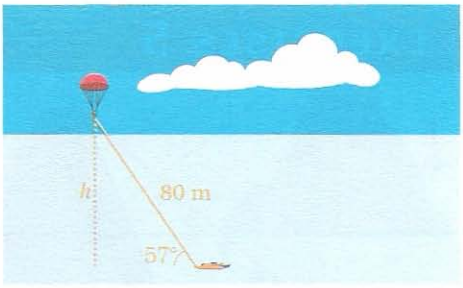
So the satellite is approximately 716.23 km from the Earth.

Check Yourself 24

1. A surveyor wishes to find the distance across a lake. He makes the measurements shown in the figure. Find the approximate distance x across the lake using this information.

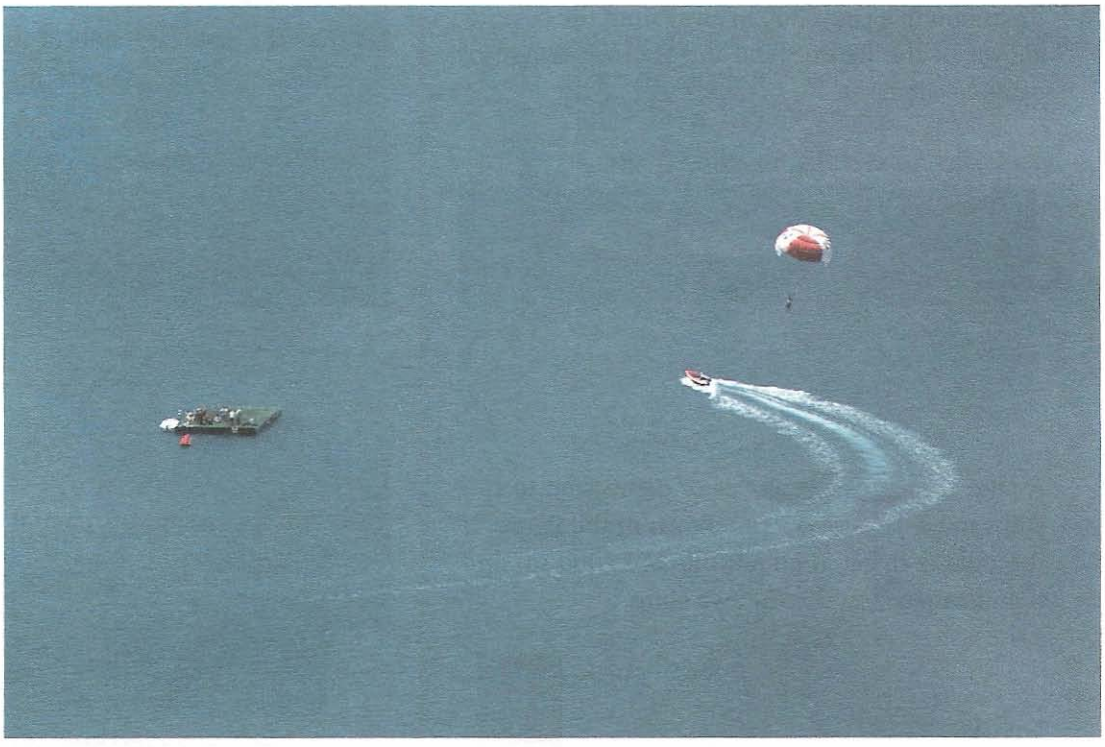


2. A parachutist is being pulled by a speedboat as shown in the figure. Find the approximate altitude h of the parachutist using the information in the figure.



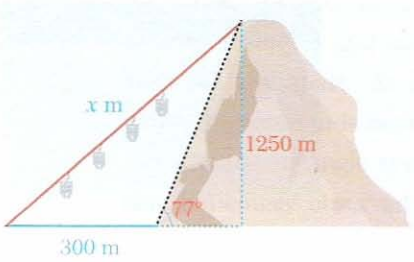
Answers

1. $x \cong 2.73$ km 2. $h \cong 67.09$ m



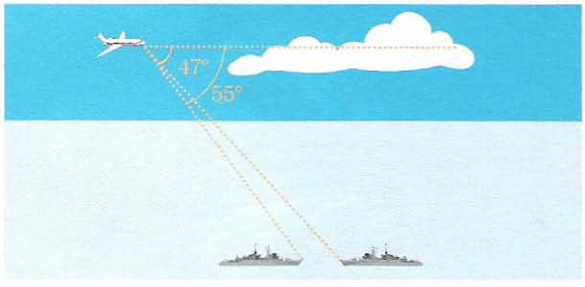
EXERCISES 5

1.



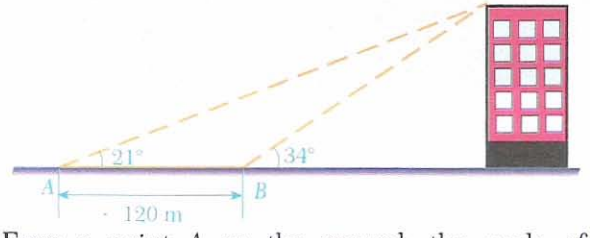
A steep mountain is inclined 77° to the horizontal and rises 1250 m above the surrounding plain. A cable car is to be installed from the top of the mountain to a point 300 m away from the base, as shown in the figure. Find the shortest length of cable needed.

2.



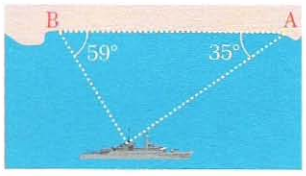
A pilot measures the angles of depression to two ships to be 47° and 55°, as shown in the figure. If the pilot is flying at an altitude of 12000 m, find the distance between the two ships.

3.

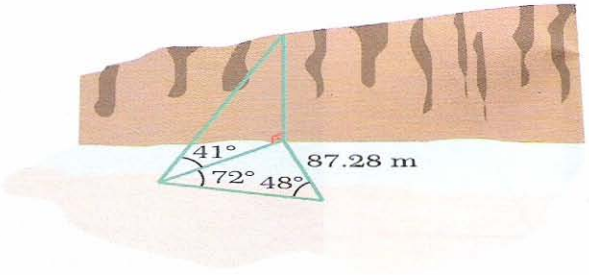


From a point A on the ground, the angle of elevation to the top of a tall building is 21°. From point B, which is 120 m closer to the building, the angle of elevation is 34°. Find the height of the building.

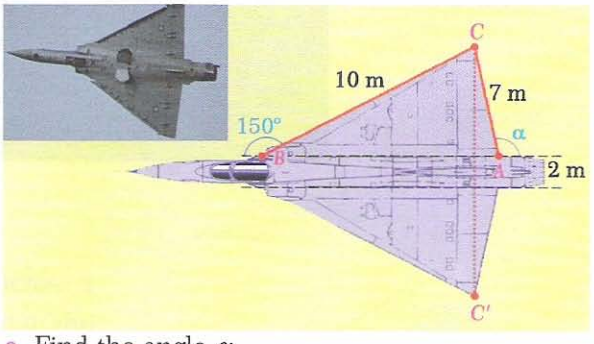
4. The boat in the figure is sailing parallel to a straight shoreline. Points A and B are 250 km apart on the shore, and $m(\angle A) = 35°$ and $m(\angle B) = 59°$. Find the shortest distance from the boat to the shore.



5. A surveyor wants to measure the height of a cliff on the opposite side of a river. She makes the measurements shown in the figure. Find the height of the cliff.



6. The figure shows an aerial view of the wings of a jet fighter.



a. Find the angle α.

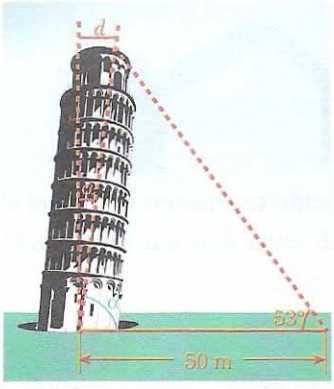
b. If the fuselage is 2 m wide, find the wing span CC'.

c. Find the area of triangle ABC.

7. ✪✪ The leaning tower of Pisa was originally perpendicular to the ground and 55 m tall. Because it is sinking into the Earth, it now leans at an angle α from the perpendicular, as shown in the figure. When the top of the tower is viewed from a point 50 m from the center of its base, the angle of elevation is 53°.

a. Find the angle α.

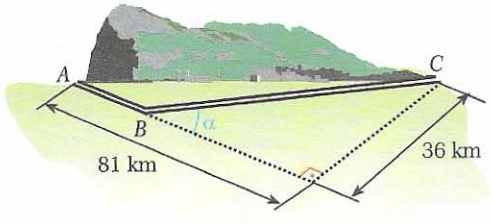
b. Find the distance d which the center of the top of the tower has moved from the perpendicular.





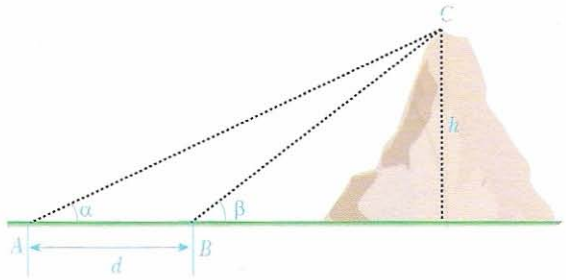
8. ✪ The figure shows a planned railroad route through three towns located at points A, B, and C. The track will turn at B toward C at an angle α.





Because of mountains between A and C, the branching point B must be at least 28 km from A. If the route is exactly 81 km long, find the measure of angle α.

9. ✪✪



To calculate the height of a mountain, angles α and β and distance d are determined, as shown in the figure.

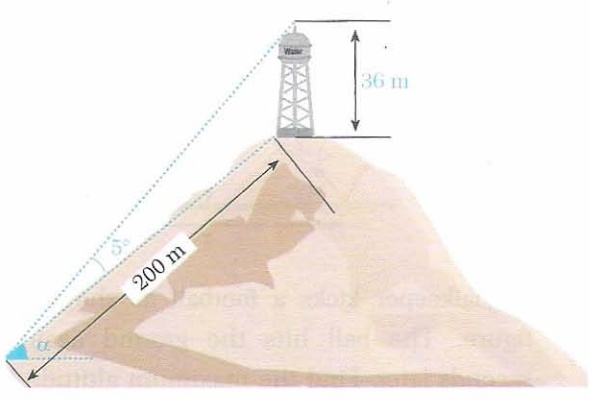
a. Find the length of BC in terms of α, β and d.

b. Show that the height h of the mountain is given by the formula

$$h = d \cdot \frac{\sin\alpha \cdot \sin\beta}{\sin(\beta - \alpha)}.$$

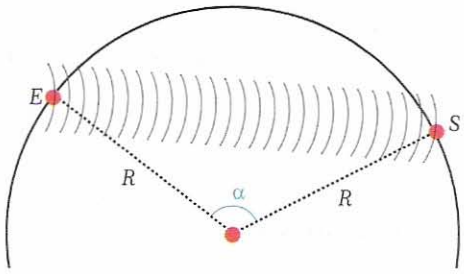
c. Use the formula in part b to find the height of the mountain if $\alpha = 26°$, $\beta = 33°$ and $d = 300$ m.

10. ✪



A water tower 36 m tall is located at the top of a hill. From a distance of 200 m down the hill, it is observed that the angle formed between the top and base of the tower is 5°. Find the angle α of inclination of the hill.

11.



E: earthquake epicenter

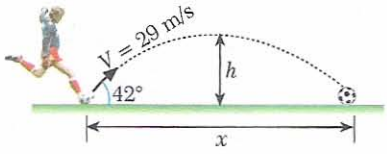
S: observation station

Seismologists study the structure of the Earth's interior by analyzing seismic waves caused by earthquakes. If the interior of the the Earth is assumed to be homogeneous, then these waves will travel in straight lines at a constant velocity v. The figure above shows a cross-sectional view of the Earth, with the earthquake epicenter at E and an observation station at S. Use the law of cosines to show that the time t for a wave to travel through the Earth's interior from E to S is given by

$$t = \frac{2R}{v} \cdot \sin\frac{\alpha}{2}$$

where R is the radius of the Earth and α is the indicated angle with its vertex at the center of the Earth.

12.

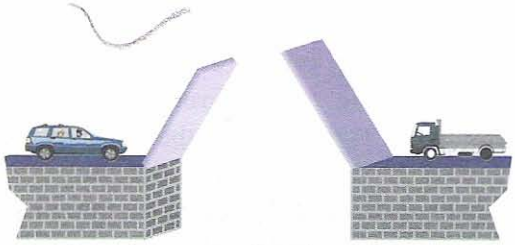


A goalkeeper kicks a football as shown in the figure. The ball hits the ground again 3.88 seconds later. Find the maximum altitude h and horizontal distance x traveled by the ball.

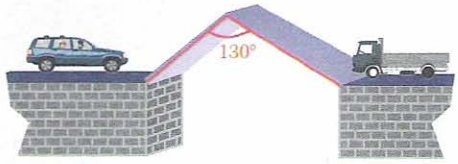
(Hint: Use $x = V_x \cdot t_{\text{total}}$, $h_{\max} = \dfrac{V_y^2}{2g}$ and $g = 10\text{m/s}^2$

where V_x and V_y are horizontal and vertical components of the initial velocity and t is time.)

13.



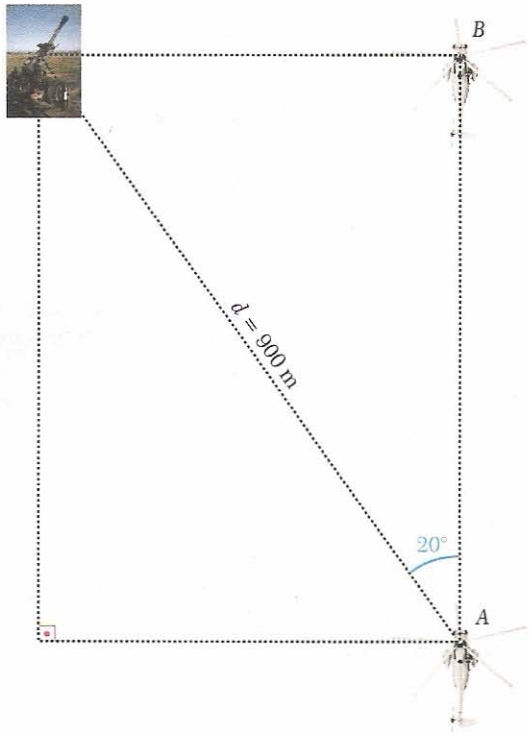
The bridge shown in the figure has two arms measuring 10 m and 12 m respectively. After deployment, the engineers realise that they have made a mistake because the ends of the arms meet at an angle of 130°.



The engineers decide to shorten the arms of the bridge. How much must they cut from the bridge to make it fit the gap?



14.

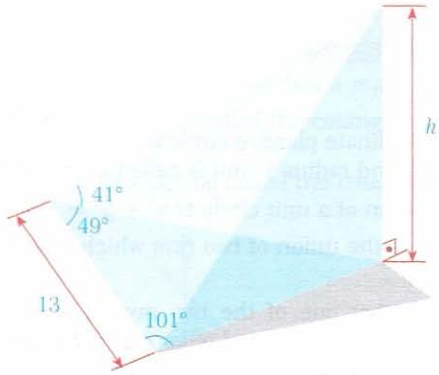


A fighter helicopter flies in a straight path AB. Its radar recognizes an enemy target located 900 m away from the helicopter. At point A, the pilot presses the fire button of a rotating machine gun and holds it for three seconds. When he stops firing, the helicopter is at point B. Calculate the approximate speed of the helicopter in meters per second.

(Hint: $V = x \cdot t$ where V is speed, x is distance and t is time.)



15. ✪✪

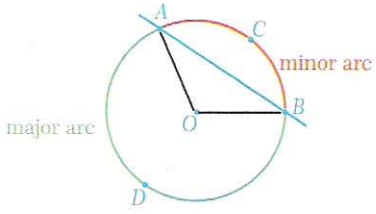


The volume V of the right triangular prism shown in the figure is $\frac{1}{3} \cdot B \cdot h$, where B is the area of the base and h is the height of the prism. Find h and V.

CHAPTER SUMMARY

Trigonometry

- In the coordinate plane, a circle which has its center C at the origin and radius 1 unit is called a **unit circle**.
- The equation of a unit circle is $x^2 + y^2 = 1$.
- An **angle** is the union of two rays which have a common endpoint.
- If we say that one of the two rays of an angle is the **initial side** of the angle and other side is the **terminal side** of the angle, then the corresponding angle is called a **directed angle**.
- If an angle is measured clockwise from its initial side then it is called a **negative angle**. If an angle is measured anticlockwise from its initial side then it is called a **positive angle**.
- The segment of a circle between the two sides of a central angle is called the **arc** corresponding to that angle. We write $\overset{\frown}{AB}$ to mean the arc corresponding to a central angle AOB.
- A line which passes through a circle is called a **secant** line.
- A line segment AB which joins two different points on a circle is called a **chord**.



- A chord which passes through the center of a circle is called a **diameter** of the circle. A diameter divides the circle into two equal arcs called **semicircles**.
- The central angle corresponding to the complete arc of a circle is called a **complete angle.**
- When the circumference of a circle is divided into 360 equal parts, the central angle corresponding to one of these arcs is called **1 degree** and denoted by **1°**.
- A degree can be divided into 60 equal parts called **minutes**, and each minute can be a divided into 60 equal parts called **seconds**. 1 minute is denoted by 1′ and 1 second is denoted by 1″.
- Let $\overset{\frown}{AB}$ be an arc of a circle with radius r such that the arc length of $\overset{\frown}{AB}$ is also r. Then the measure of the central angle corresponding to $\overset{\frown}{AB}$ is called **1 radian**. It is denoted by **1 rad** or 1^R.
- The measure of the central angle corresponding to the complete arc of a circle is **2π radians**.
- On a unit circle, the measure of an angle in radians is equal to the length of the corresponding arc.
- $\frac{D}{180°} = \frac{R}{\pi^R}$ where D and R represent degree and radian measurements respectively.
- An angle in the coordinate plane whose vertex is at the origin and whose initial side lies along the positive x-axis is said to be in **standard position**.
- Two or more angles whose terminal sides coincide with each other when they are in standard position are called **coterminal** angles.
- Let β be an angle which is greater than 360°. Then the positive angle $\alpha \in [0, 360°)$ which is coterminal with β is called the **primary directed angle** of β.
- The positive real number $t \in [0, 2\pi)$ which differs from a real number by integral multiples of 2π is called a **primary directed arc.**

Trigonometric Ratios

Ratio name	Ratio abbreviation	Ratio definition	Ratio abbreviation
sine	$\sin\theta$	$\frac{\text{length of side opposite } \theta}{\text{length of hypotenuse}}$	$\frac{opp}{hyp}$
cosine	$\cos\theta$	$\frac{\text{length of side adjacent to } \theta}{\text{length of hypotenuse}}$	$\frac{adj}{hyp}$
tangent	$\tan\theta$	$\frac{\text{length of side opposite } \theta}{\text{length of side adjacent to } \theta}$	$\frac{opp}{adj}$
cotangent	$\cot\theta$	$\frac{\text{length of side adjacent to } \theta}{\text{length of side opposite } \theta}$	$\frac{adj}{opp}$
secant	$\sec\theta$	$\frac{\text{length of hypotenuse}}{\text{length of side adjacent to } \theta}$	$\frac{hyp}{adj}$
cosecant	$\csc\theta$	$\frac{\text{length of hypotenuse}}{\text{length of side opposite } \theta}$	$\frac{hyp}{opp}$

- Trigonometric ratios of special acute angles:

θ in degrees	θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$

- Fundamental trigonometric identities:

$\sin^2\theta + \cos^2\theta = 1$ $\quad \tan^2\theta + 1 = \sec^2\theta$

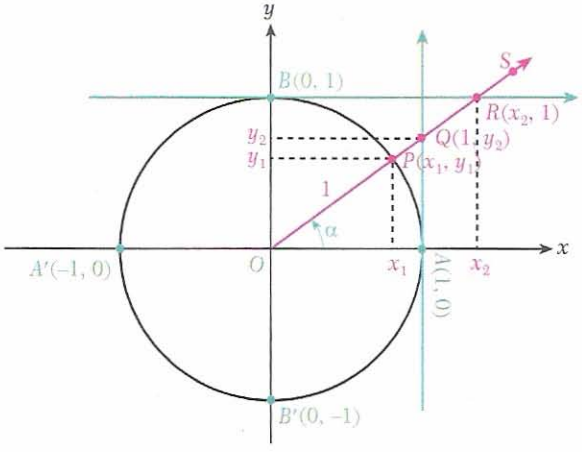
$\cot^2\theta + 1 = \csc^2\theta$ $\quad \tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ $\quad \tan\theta \cdot \cot\theta = 1$

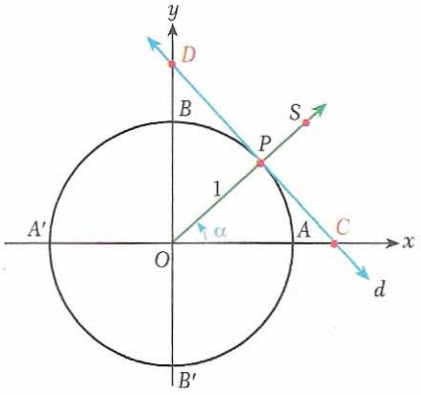
$\csc\theta = \dfrac{1}{\sin\theta}$ $\quad \csc\theta = \dfrac{1}{\sin\theta}$

$\csc\theta = \dfrac{1}{\sin\theta}$ $\quad \sec\theta = \dfrac{1}{\cos\theta}$

Trigonometric Functions



- The ordinate (y-coordinate) of the point P on the unit circle is called the **sine of angle α**. It is denoted by **sin α**. The function which matches a real number α to the real number sin α is called the **sine function**.
- The y-axis can also be called the sine axis.
- The abscissa (x-coordinate) of the point P on the unit circle is called the **cosine of angle α**. It is denoted by **cos α**. The function which matches a real number α to the real number cos α is called the **cosine function**.
- The x-axis can also be called the **cosine axis**.
- The ordinate (y-coordinate) of the point Q on the line $x = 1$ is called the **tangent of angle α**. It is denoted by **tan α**. The function which matches a real number α to the real number tan α is called the **tangent function**.
- The line $x = 1$ can also be called the **tangent axis**.
- The abscissa (x-coordinate) of the point R on the line $y = 1$ is called the **cotangent of angle α**. It is denoted by **cot α**. The function which matches a real number α to the real number cot α is called the **cotangent function**.
- The line $y = 1$ can also be called the **cotangent axis**.



- The abscissa (x-coordinate) of the point C on the x-axis is called the **secant of angle α**. It is denoted by **sec α**. The function which matches a real number α to the real number sec α is called the **secant function**.
- The ordinate (y-coordinate) of point D on the y-axis is called the **cosecant of angle α**. It is denoted by **csc α**. The function which matches a real number α to the real number csc α is called the **cosecant function**.
- If the terminal side of an angle coincides with a coordinate axis then the angle is called a **quadrantal angle**.
- Trigonometric values of quadrantal angles:

θ in degrees	θ in radians	$\sin\theta$ (y)	$\cos\theta$ (x)	$\tan\theta$ (y/x)	$\cot\theta$ (x/y)	$\sec\theta$ ($1/x$)	$\csc\theta$ ($1/y$)
0°	0	0	1	0	undefined	1	undefined
90°	$\frac{\pi}{2}$	1	0	undefined	0	undefined	1
180°	π	0	−1	0	undefined	−1	undefined
270°	$\frac{3\pi}{2}$	−1	0	undefined	0	undefined	−1
360°	2π	0	1	0	undefined	1	undefined

- The positive acute angle α which is formed by the terminal side of a nonquadrantal angle θ and the x-axis is called the **reference angle** for θ.

Trigonometric Theorems and Formulas

1. **The law of cosines**: In a triangle ABC with side lengths a, b and c,

$a^2 = b^2 + c^2 - 2bc \cdot \cos A$

$b^2 = a^2 + c^2 - 2ac \cdot \cos B$

$c^2 = a^2 + b^2 - 2ab \cdot \cos C.$

Rearranging these results gives

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

2. The **law of sines**: In a triangle ABC with side lengths a, b, c which is inscribed in a circle with radius R,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

3. Area of a triangle: In a triangle ABC with sides a, b and c, the following properties hold

$$A(\Delta ABC) = \frac{1}{2} \cdot a \cdot b \cdot \sin C$$

$$A(\Delta ABC) = \frac{1}{2} \cdot a \cdot c \cdot \sin B$$

$$A(\Delta ABC) = \frac{1}{2} \cdot b \cdot c \cdot \sin A.$$

4. **Heron's Formula**: Let ABC be a triangle with sides a, b, c, and perimeter $a + b + c = 2u$. Then

$$A(\Delta ABC) = \sqrt{u(u-a)(u-b)(u-c)}.$$

5. Area of an inscribed triangle: Let ABC be a triangle with sides a, b, c which is inscribed in a circle with radius R. Then $A(\Delta ABC) = \frac{a \cdot b \cdot c}{4R}$.

6. Sum and difference formulas:

$\sin(x + y) = (\sin x \cdot \cos y) + (\cos x \cdot \sin y)$

$\sin(x - y) = (\sin x \cdot \cos y) - (\cos x \cdot \sin y)$

$\cos(x + y) = (\cos x \cdot \cos y) - (\sin x \cdot \sin y)$

$\cos(x - y) = (\cos x \cdot \cos y) + (\sin x \cdot \sin y)$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - (\tan x \cdot \tan y)}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + (\tan x \cdot \tan y)}$$

$$\cot(x+y) = \frac{(\cot x \cdot \cot y) - 1}{\cot y + \cot x}$$

$$\cot(x-y) = \frac{(\cot x \cdot \cot y) + 1}{\cot y - \cot x}$$

7. Double-angle formulas:

$\sin 2x = 2\sin x \cdot \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x = 1 - 2\sin^2 x$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$\cot 2x = \frac{\cot^2 x - 1}{2\cot x}$$

8. Reduction formulas:

Reduction Formulas for the First Quadrant $\left(0 < \alpha < \frac{\pi}{2}\right)$

$\sin(0 + \alpha) = \sin\alpha$	$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$
$\sin(0 + \alpha) = \sin\alpha$	$\cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$
$\tan(0 + \alpha) = \tan\alpha$	$\tan\left(\frac{\pi}{2} - \alpha\right) = \cot\alpha$
$\cot(0 + \alpha) = \cot\alpha$	$\cot\left(\frac{\pi}{2} - \alpha\right) = \tan\alpha$

Reduction Formulas for the Second Quadrant $\left(\frac{\pi}{2} < \alpha < \pi\right)$

$\sin(\pi - \alpha) = \sin\alpha$	$\sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$
$\cos(\pi - \alpha) = -\cos\alpha$	$\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha$
$\tan(\pi - \alpha) = -\tan\alpha$	$\tan\left(\frac{\pi}{2} + \alpha\right) = -\cot\alpha$
$\cot(\pi - \alpha) = -\cot\alpha$	$\cot\left(\frac{\pi}{2} + \alpha\right) = -\tan\alpha$

Reduction Formulas for the Third Quadrant

$\left(\pi < \alpha < \frac{3\pi}{2}\right)$

$\sin(\pi + \alpha) = -\sin\alpha$	$\sin\left(\frac{3\pi}{2} - \alpha\right) = -\cos\alpha$
$\cos(\pi + \alpha) = -\cos\alpha$	$\cos\left(\frac{3\pi}{2} - \alpha\right) = -\sin\alpha$
$\tan(\pi + \alpha) = \tan\alpha$	$\tan\left(\frac{3\pi}{2} - \alpha\right) = \cot\alpha$
$\cot(\pi + \alpha) = \cot\alpha$	$\cot\left(\frac{3\pi}{2} - \alpha\right) = \tan\alpha$

Reduction Formulas for the Fourth Quadrant

$\left(\frac{3\pi}{2} < \alpha < 2\pi\right)$

$\sin(2\pi - \alpha) = -\sin\alpha$	$\sin\left(\frac{3\pi}{2} + \alpha\right) = -\cos\alpha$
$\cos(2\pi - \alpha) = \cos\alpha$	$\cos\left(\frac{3\pi}{2} + \alpha\right) = \sin\alpha$
$\tan(2\pi - \alpha) = -\tan\alpha$	$\tan\left(\frac{3\pi}{2} - \alpha\right) = -\cot\alpha$
$\cot(2\pi - \alpha) = -\cot\alpha$	$\cot\left(\frac{3\pi}{2} + \alpha\right) = -\tan\alpha$

9. Sum to product formulas:

$$\sin a + \sin b = 2 \cdot \sin\frac{a+b}{2} \cdot \cos\frac{a-b}{2}$$

$$\sin a - \sin b = 2 \cdot \cos\frac{a+b}{2} \cdot \sin\frac{a-b}{2}$$

$$\cos a + \cos b = 2 \cdot \cos\frac{a+b}{2} \cdot \cos\frac{a-b}{2}$$

$$\cos a - \cos b = -2 \cdot \sin\frac{a+b}{2} \cdot \sin\frac{a-b}{2}$$

$$\tan a + \tan b = \frac{\sin(a+b)}{\cos a \cdot \cos b}$$

$$\tan a - \tan b = \frac{\sin(a-b)}{\cos a \cdot \cos b}$$

$$\cot a + \cot b = \frac{\sin(a+b)}{\sin a \cdot \sin b}$$

$$\cot a - \cot b = -\frac{\sin(a-b)}{\sin a \cdot \sin b}$$

10. Product to sum formulas:

$$\sin x \cdot \sin y = -\frac{1}{2}[\cos(x+y) - \cos(x-y)]$$

$$\cos x \cdot \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\sin x \cdot \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\cos x \cdot \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$

Concept Check

1. What is an angle? Name three different units of angle measure.
2. What do we mean when we talk about the direction of an angle?
3. How do we measure very small angles?
4. Why are trigonometric identities useful?
5. What is the relation between a trigonometric ratio and trigonometric function?
6. Are all of the trigonometric functions defined for all real numbers? What are the exceptions?
7. What is a reference angle? Why is it useful?
8. What are reduction formulas? When are they useful?
9. When are the reduction formulas more effective than using a reference angle?
10. Give three examples of how trigonometry can be applied to real-world problems.



CHAPTER REVIEW TEST 1

1. $(a-3)x^2 + (b+1)y^2 = 1$ is the equation of a unit circle. Find (a, b).

A) $(2, 1)$ B) $(1, 2)$ C) $(3, 2)$ D) $(3, 4)$ E) $(4, 0)$

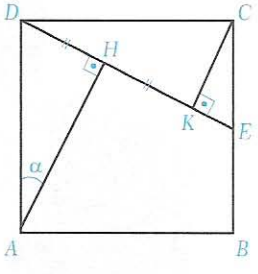
2. $P\left(\frac{\sqrt{3}}{2}, b\right)$ is a point on the unit circle. Given that P is in the fourth quadrant, find b.

A) $-\frac{1}{2}$ B) $-\frac{\sqrt{3}}{2}$ C) -1 D) $\frac{1}{2}$ E) $\frac{\sqrt{3}}{2}$

3. $\sin x - \cos x = \frac{1}{4}$ is given. What is $\sin x \cdot \cos x$?

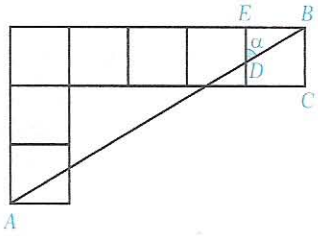
A) $\frac{15}{16}$ B) $\frac{15}{32}$ C) $\frac{3}{4}$ D) $\frac{3}{8}$ E) $\frac{7}{8}$

4. In the figure, $ABCD$ is a square with $m(\angle CKE) = 90°$, $m(\angle DHA) = 90°$, $DH = HK$ and $m(\angle DAH) = \alpha$. Find $\tan \alpha$.



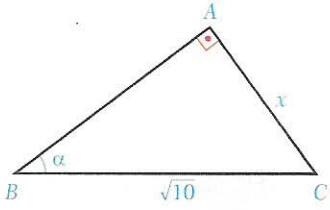
A) $\frac{\sqrt{7}}{2}$ B) $\frac{\sqrt{3}}{2}$ C) $\frac{3}{4}$ D) $\frac{2}{3}$ E) $\frac{1}{2}$

5. This figure contains seven identical squares. What is $\tan(\angle EDB)$?



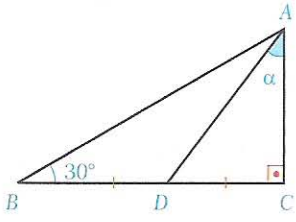
A) $\frac{3}{4}$ B) $\frac{4}{3}$ C) $\frac{5}{4}$ D) $\frac{3}{5}$ E) $\frac{5}{3}$

6. In the figure, $BC = \sqrt{10}$ and $\tan \alpha = \frac{1}{3}$. Find $AC = x$.



A) $3\sqrt{3}$ B) 3 C) $\sqrt{3}$ D) $\frac{1}{\sqrt{3}}$ E) 1

7. In the figure, $m(\angle ABC) = 30°$, $m(\angle BCA) = 90°$ and $DB = DC$. What is $\tan(\angle DAC)$?



A) $\frac{\sqrt{3}}{2}$ B) $\frac{\sqrt{5}}{2}$ C) $\frac{\sqrt{5}}{3}$ D) $\sqrt{3}$ E) $2\sqrt{3}$

8. Which one of the following is equivalent to $\sin 40°$?

A) $\sin 220°$ B) $\cos 140°$ C) $\sin 50°$ D) $\sin(-40°)$ E) $\cos(-50°)$

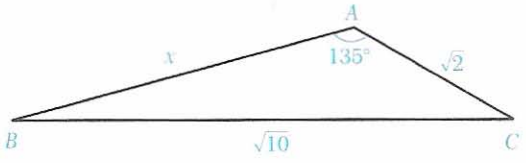
9. Which one of the following is <u>not</u> equivalent to $\sin\left(\frac{\pi}{2}-\alpha\right)$?

A) $\sin\left(\frac{\pi}{2}+\alpha\right)$ B) $\cos(2\pi-\alpha)$ C) $\cos(-\alpha)$ D) $\cos\alpha$ E) $\sin(-\alpha)$

10. $\frac{3}{\cos x}=\frac{4}{\sin x}$ is given. Find the positive value of $\cos x$.

A) $\frac{2}{3}$ B) $\frac{2}{5}$ C) $\frac{3}{5}$ D) $\frac{4}{5}$ E) $\frac{\sqrt{3}}{5}$

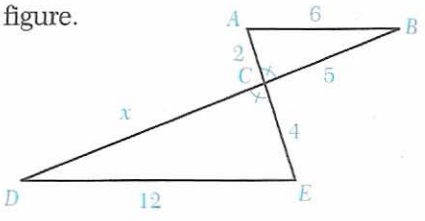
11.



Find x in the figure.

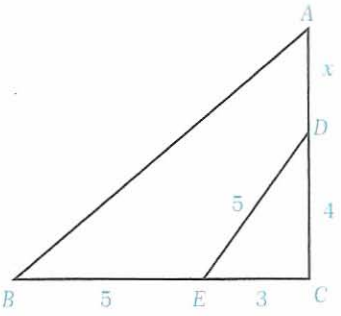
A) 1 B) $\sqrt{2}$ C) 2 D) $2\sqrt{2}$ E) 4

12. Find x in the figure.



A) 5 B) 6 C) 8 D) 10 E) 12

13. In the figure, ABC is a triangle with $AB = 10$ cm, $DE = BE = 5$ cm, $EC = 3$ cm and $DC = 4$ cm. Find $AD = x$.

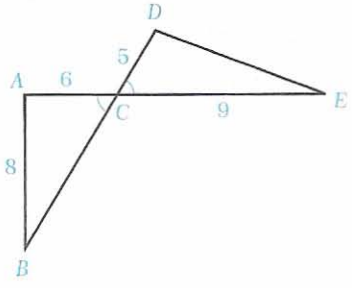


A) 2 B) 4 C) 6 D) 8 E) 10

14. ABC is a triangle such that $A(\Delta ABC)=\frac{15\sqrt{2}}{2}$ cm^2, $a = 5$ cm and $c = 6$ cm. Which one of the following is a possible value of $m(\angle B)$?

A) 150° B) 135° C) 120° D) 90° E) 60°

15. In the figure, $AB \perp AE$, $AB = 8$ cm, $AC = 6$ cm, $CE = 9$ cm and $CD = 5$ cm. Find $A(\Delta CDE)$.



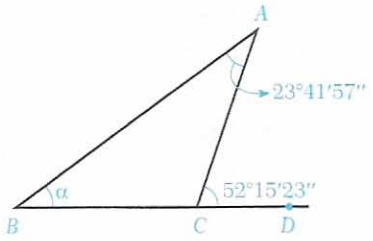
A) $\frac{54}{5}$ cm^2 B) 12 cm^2 C) 18 cm^2 D) $\frac{72}{5}$ cm^2 E) 15 cm^2

16. An inscribed triangle has side lengths of 3, 7 and 8 units respectively. Find the circumference of its circumscribed circle.

A) $7\sqrt{3}\pi$ units B) $\frac{7}{3}\pi$ units C) $\frac{14\sqrt{3}}{3}\pi$ units D) $\frac{7\sqrt{3}}{7}\pi$ units E) $\frac{3\sqrt{3}}{7}\pi$ units

CHAPTER REVIEW TEST 2

1. Find $m(\angle B) = \alpha$ in the figure.



A) 28° 34′ 26″ B) 27° 33′ 26″

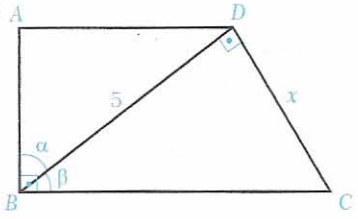
C) 27° 34′ 26″ D) 28° 33′ 26″

E) 29° 33′ 26″

2. Find the primary directed angle of −3333°.

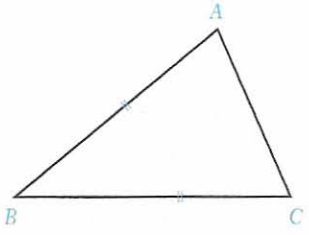
A) 277° B) 267° C) 263° D) 257° E) 253°

3. In the figure, $ABCD$ is a trapezoid and α and β are complementary angles, $BD = 5$ and $\tan\alpha = \frac{4}{3}$. Find $DC = x$.



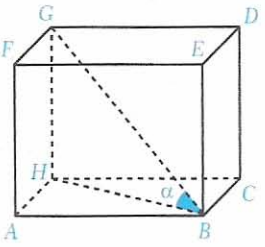
A) $\frac{5}{12}$ B) $\frac{4}{15}$ C) $\frac{12}{5}$ D) $\frac{15}{4}$ E) $\frac{20}{3}$

4. In the figure, $AB = BC$ and $\cos B = \frac{3}{5}$. Find $\cot C$.



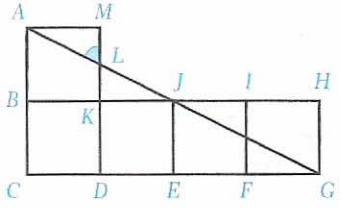
A) 2 B) $\frac{1}{2}$ C) $\frac{3}{4}$ D) $\frac{4}{3}$ E) $\frac{5}{4}$

5. In the figure, $ABCDEFGH$ is a cube. Find $\tan(\angle GBH)$.



A) $\frac{1}{\sqrt{3}}$ B) $\frac{\sqrt{2}}{\sqrt{3}}$ C) $\frac{1}{\sqrt{2}}$ D) $\frac{\sqrt{3}}{\sqrt{2}}$ E) $\sqrt{3}$

6. The figure contains five identical squares. What is $\sin(\angle ALM)$?



A) $\frac{1}{2}$ B) $\frac{\sqrt{3}}{2}$ C) $\frac{1}{\sqrt{5}}$ D) $\frac{2}{\sqrt{5}}$ E) $\frac{\sqrt{3}}{5}$

7. $\pi < \theta < \frac{3\pi}{2}$ and $\sin\theta = -\frac{1}{4}$ are given. What is $\tan\theta - \sec\theta$?

A) $\frac{\sqrt{15}}{3}$ B) $\frac{\sqrt{15}}{5}$ C) $\frac{\sqrt{15}}{15}$ D) $-\frac{\sqrt{15}}{5}$ E) $-\frac{\sqrt{15}}{3}$

8. $\frac{\pi}{4} < \beta < \frac{\pi}{2}$ and $\sin 2\beta = \frac{12}{13}$ are given. What is $\tan\beta + \cot\beta$?

A) $\frac{1}{13}$ B) $\frac{1}{13}$ C) $\frac{24}{13}$ D) $\frac{6}{13}$ E) $\frac{13}{6}$

9. Which expression is equivalent to

$\cos\left(\frac{3\pi}{2}+x\right)+\sin\left(\frac{17\pi}{2}-x\right)+\tan(2\pi-x)$?

A) $\cos x + \sin x + \tan x$

B) $\cos x - \sin x - \tan x$

C) $-\cos x + \sin x - \tan x$

D) $-\cos x - \sin x + \tan x$

E) $\cos x + \sin x - \tan x$

10. Which one of the following is <u>false</u>?

A) $\sin 40° = \cos 50°$

B) $\tan 210° = \cot 30°$

C) $\tan 45° = \cot 225°$

D) $\sec 89° = \csc 1°$

E) $\sin 280° = -\cos 10°$

11. Find the value of $\frac{\sin 15°}{\sin 45°}+\frac{\cos 15°}{\cos 45°}$.

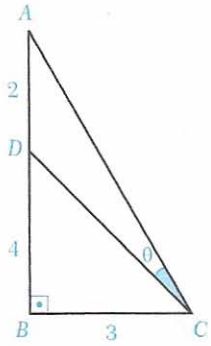
A) $-\sqrt{3}$ B) $-\frac{\sqrt{3}}{2}$ C) 0 D) $\frac{\sqrt{3}}{2}$ E) $\sqrt{3}$

12. $x \in [0, \pi]$, $y \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, $\cos x = -\frac{1}{3}$ and $\sin y = \frac{1}{2}$ are given. Find $\cos(x + y)$.

A) $\frac{-\sqrt{3}-2\sqrt{2}}{6}$ B) $\frac{\sqrt{3}+2\sqrt{2}}{6}$ C) $\frac{\sqrt{3}-2\sqrt{2}}{6}$

D) $\frac{1}{6}$ E) $\frac{2\sqrt{2}-\sqrt{3}}{6}$

13. In the figure, $m(\angle ABC) = 90°$, $AD = 2$, $DB = 4$ and $BC = 3$. Find $\sin\theta$.



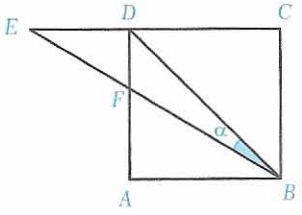
A) $\frac{2}{5\sqrt{5}}$ B) $\frac{11}{5\sqrt{5}}$ C) $\frac{2}{11}$ D) $\frac{11}{2}$ E) $\frac{2}{5}$

14. $a+b=\frac{\pi}{3}$ is given.

Evaluate $(\sin a - \sin b)^2 + (\cos a + \cos b)^2$.

A) 0 B) 1 C) 2 D) 3 E) 4

15. In the figure, $ABCD$ is a square and $3 \cdot DE = AB$. Find $\csc\alpha$.



A) $\frac{1}{5\sqrt{2}}$ B) $\frac{5\sqrt{2}}{7}$ C) $5\sqrt{2}$ D) $\frac{7}{5\sqrt{2}}$ E) 7

16. $\cos x = \frac{1}{3}$ is given. If x is an acute angle, find $\sin 2x$.

A) $\frac{2\sqrt{2}}{2}$ B) $\frac{2\sqrt{2}}{3}$ C) $\frac{\sqrt{2}}{9}$ D) $\frac{4\sqrt{2}}{9}$ E) $\frac{8\sqrt{2}}{9}$

CHAPTER REVIEW TEST 3

1. Find the primary directed angle of $-\frac{72\pi}{5}$.

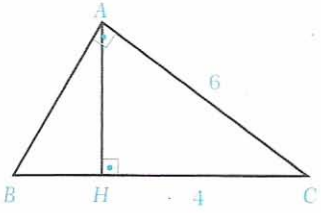
A) $\frac{2\pi}{5}$ B) $\frac{3\pi}{5}$ C) $\frac{4\pi}{5}$ D) $\frac{7\pi}{5}$ E) $\frac{8\pi}{5}$

2. Find the coordinates of the terminal point of an arc in standard position with length $\frac{3\pi}{4}$ on the unit circle.

A) $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ B) $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ C) $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

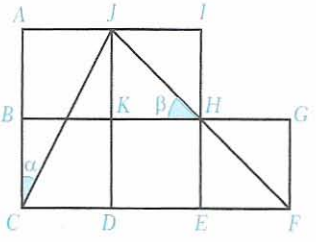
D) $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ E) $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

3. In the figure, ABC is a right triangle with altitude AH. $CH = 4$ and $AC = 6$ are given. Find $\sin B$.



A) $\frac{3}{4}$ B) $\frac{2}{3}$ C) $\frac{1}{2}$ D) $\frac{1}{3}$ E) $\frac{1}{4}$

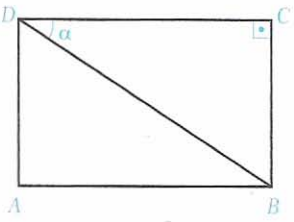
4. The figure contains five identical squares. What is $\sin\alpha + \cos\beta$?



A) $\frac{\sqrt{2}+\sqrt{5}}{\sqrt{10}}$ B) $\frac{1+\sqrt{5}}{\sqrt{10}}$ C) $\frac{\sqrt{3}+\sqrt{5}}{\sqrt{10}}$

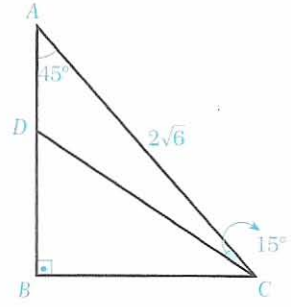
D) $\frac{1+\sqrt{2}}{\sqrt{10}}$ E) $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{10}}$

5. In the figure, $ABCD$ is rectangle with $AB = 3 \cdot AD$. What is a possible value of $\tan\alpha$?



A) $\frac{1}{\sqrt{10}}$ B) $\frac{3}{\sqrt{10}}$ C) $\frac{1}{3}$ D) $\frac{1}{6}$ E) $\frac{\sqrt{10}}{3}$

6. In the figure, ABC is a right triangle with $m(\angle B) = 90°$, $m(\angle BAC) = 45°$, $m(\angle ACD) = 15°$ and $AC = 2\sqrt{6}$. Find DC.



A) 4 B) $2\sqrt{3}$ C) $\sqrt{6}$ D) 2 E) 5

7. $0 < x < \pi$ and $\cos x = -\frac{3}{5}$ are given. What is $\csc x - \sec x + \tan x$?

A) $\frac{19}{12}$ B) $\frac{17}{12}$ C) $\frac{7}{12}$ D) $-\frac{13}{6}$ E) $-\frac{7}{6}$

8. $\cos 25° = a$ is given. Find $\sin 205°$ in terms of a.

A) $-a$ B) $-\frac{1}{a}$ C) $\sqrt{1-a^2}$

D) $-\sqrt{1-a^2}$ E) a

9. $\cos 36° \cong 0.8090$ and $\cos 37° \cong 0.7986$ are given. Find $\cos(36° 15')$.

A) 0.8116 B) 0.8074 C) 0.8064 D) 0.8044 E) 0.8038

10. $\sin 36° \cong 0.5878$ and $\tan 42° \cong 0.9004$ are given. Find $(3 \cdot \cos 54°) - (2 \cdot \cot 48°)$.

A) 0.0374 B) 0.0274 C) −0.0174 D) −0.0274 E) −0.0374

11. $\cos\alpha = \frac{4}{5}$ is given. If α is an acute angle, what is $\sin\frac{\alpha}{2}$?

A) $-\frac{1}{\sqrt{10}}$ B) $-\frac{4}{\sqrt{5}}$ C) $\frac{3}{5}$ D) $\frac{4}{\sqrt{10}}$ E) $\frac{1}{\sqrt{10}}$

12. $\sin 25° = a$ is given. Find the value of $\sin 40°$ in terms of a.

A) $2a^2$ B) $2a^2 - 1$ C) $1 - 2a^2$ D) $\frac{1-2a^2}{a^2+1}$ E) $2a^2 + 1$

13. Which one of the following is equivalent to $\frac{\sin a + \sin 3a + \sin 5a}{\cos a + \cos 3a + \cos 5a}$?

A) $\tan 3a$ B) $\cot 3a$ C) $\cot 4a$ D) $\tan 6a$ E) $\tan 8a$

14. $6x = \pi$ is given. Which one of the following is equivalent to $\frac{\tan 5x + \tan x}{\tan 5x - \tan x}$?

A) 1 B) −1 C) 0 D) $3x$ E) $2x$

15. $a = \frac{\pi}{20}$ is given. Which one of the following is equivalent to $\frac{\sin 2a - \sin 4a}{\sin 7a \cdot \cos 9a}$?

A) −3 B) −2 C) −1 D) 2 E) 3

16. What is $\frac{1}{\cos 195°} + \frac{1}{\sin 195°}$?

A) $-2\sqrt{2}$ B) $-4\sqrt{2}$ C) $-2\sqrt{6}$ D) $-4\sqrt{6}$ E) $-6\sqrt{2}$

ANSWERS TO EXERCISES

EXERCISES 1

1. a. (0, 3) b. (2, –2) c. (0, 1) d. (0, 0) 2. a. $y = \pm\frac{\sqrt{3}}{2}$ b. $y = \pm\frac{1}{2}$ c. $x = \pm\frac{2\sqrt{2}}{3}$ d. $x = \pm\frac{\sqrt{2}}{2}$
3. a. initial side BA, terminal side BC b. initial side EF, terminal side ED c. initial side HG, terminal side HI
d. initial side KJ, terminal side KL 4. a. positive b. negative c. positive 5. a. negative angle AOB b. positive angle MON
c. positive angle ROP d. negative angle TOS 6. a. minor arc $\overset{\frown}{ACB}$, major arc $\overset{\frown}{BDA}$ b. minor arc $\overset{\frown}{KML}$, major arc $\overset{\frown}{KNL}$
7. a. –270° b. 240° c. –1200° d. 810° 8. a. $-\frac{\pi}{2}$ b. $\frac{6\pi}{5}$ c. $-\frac{9\pi}{2}$ d. $\frac{20\pi}{3}$ 9. a. 10.75° b. 80.25°
c. $37.358\overline{3}°$ d. 90° 10. a. 10° 06′ b. 82° 09′ c. 54° 18′ d. 23° 43′ 48″ 11. a. 102° 50′ 45″ b. 23° 49′ 48″
c. –105° 22′ 56″ d. 247° 52′ 04″ 12. a. $\frac{\pi}{3}^R$ b. 135° c. $\frac{7\pi}{6}^R$ d. 330° 13. a. $\frac{5\pi}{9}$ b. $\frac{25\pi}{18}$ c. $\frac{20\pi}{3}$
d. $-\frac{50\pi}{3}$ e. 15° f. 170° g. 9045° h. –510° 14. a. 154° b. 359° c. $\frac{10\pi}{9}$ d. $\frac{24\pi}{19}$ 15. a. (–1, 0)
b. $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ 16. 108° 17. a. $x = 105° + (k \cdot 180°)$ b. $x = \pi + 24k\pi$ c. $x = k \cdot 90°$ d. $x = \frac{\pi}{3} + \frac{k \cdot 2\pi}{3}$

EXERCISES 2

1. a. $\sqrt{17}$ b. $\frac{\sqrt{2}}{6}$ c. $\frac{\sqrt{34}}{6}$ 2. $\sqrt{91}$ 3. 6 4. $8 + 2\sqrt{3}$ 5. 13 6. a. $\frac{9}{4}$ b. 12 7. $\frac{729}{32}$ units 8. a. $\sec x$ b. 1 c. $\csc x \cdot \sec x$
d. $\sin x$ e. $\sin x$ f. $\cot x$ 9. a. $1 - \sin x$ b. $1 - \sin x$ c. $\sin^2 x$ d. $\cot x$ e. 1 f. $(\sec x + 1)^2$ 12. a. 1 b. 3 13. 2

EXERCISES 3

3. a. not on the unit circle b. not on the unit circle 4. a. $z > y > x$ b. $z > x > y > p$ c. $x < y < z < p$ 6. a. 0
b. 1 7. a. 12° b. 68° c. 32° d. 48° e. 50° f. 30° g. 70° h. 10° 8. a. 25° b. 40° c. 65° d. 55° e. 5° f. 85° g. 80°
h. 40° 9. a. 80° b. 36° c. 60° d. 30° e. 20° f. 51° g. 80° h. 5° 10. a. $\frac{\pi}{11}$ b. $\frac{5\pi}{12}$ c. $\frac{5\pi}{13}$ d. $\frac{3\pi}{14}$ e. $\frac{2\pi}{13}$ f. $\frac{2\pi}{5}$
g. $\frac{5\pi}{17}$ h. $\frac{2\pi}{19}$ 11. a. $\frac{\pi}{8}$ b. $\frac{3\pi}{10}$ c. $\frac{\pi}{12}$ d. $\frac{5\pi}{14}$ e. $\frac{\pi}{6}$ f. $\frac{\pi}{9}$ g. $\frac{5\pi}{12}$ h. $\frac{2\pi}{15}$ 12. a. $\frac{\pi}{6}$ b. $\frac{\pi}{3}$ c. $\frac{3\pi}{13}$ d. $\frac{4\pi}{15}$ e. $\frac{\pi}{7}$
f. $\frac{15\pi}{73}$ g. $\frac{\pi}{7}$ h. $\frac{19\pi}{99}$ 14. a. $\frac{\sqrt{3}}{2}$ b. $-\frac{1}{2}$ c. –1 d. –1 e. $\frac{\sqrt{2}}{2}$ f. $\frac{1}{2}$ g. 1 h. $-\sqrt{3}$ 15. a. $\frac{\sqrt{2}}{2}$ b. $-\frac{\sqrt{3}}{2}$

c. $\sqrt{3}$ d. $-\sqrt{3}$ e. $-\frac{1}{2}$ f. $\frac{\sqrt{2}}{2}$ g. -1 h. $-\frac{\sqrt{3}}{3}$ 16. a. $\frac{1}{2}$ b. $\frac{1}{2}$ c. 1 d. $-\sqrt{3}$ e. $-\frac{\sqrt{2}}{2}$ f. $-\frac{1}{2}$ g. -1 h. $-\sqrt{3}$

17. a. $-\frac{1}{2}$ b. $-\frac{1}{2}$ c. -1 d. $\sqrt{3}$ e. $-\frac{\sqrt{2}}{2}$ f. $-\frac{1}{2}$ g. 1 h. $\sqrt{3}$ 18. a. $\sqrt{2}$ b. 0 c. 6 19. a. $\tan\alpha + \cot\alpha - 2\sin\alpha$ b. 0

20. a. $\cos\theta = \frac{1}{2}$, $\tan\theta = \sqrt{3}$, $\cot\theta = \frac{\sqrt{3}}{3}$ b. $\sin\theta = \frac{\sqrt{2}}{2}$, $\tan\theta = -1$, $\cot\theta = -1$ c. $\sin\theta = -\frac{5}{\sqrt{41}}$, $\cos\theta = -\frac{4}{\sqrt{41}}$, $\cot\theta = \frac{4}{5}$

d. $\sin\theta = -\frac{3}{\sqrt{13}}$, $\cos\theta = \frac{2}{\sqrt{13}}$, $\tan\theta = -\frac{3}{2}$ 21. a. $\cos\theta = \frac{\sqrt{2}}{2}$, $\tan\theta = -1$, $\cot\theta = -1$

b. $\sin\theta = -\frac{7}{25}$, $\tan\theta = \frac{7}{24}$, $\cot\theta = \frac{24}{7}$ c. $\sin\theta = \frac{4}{\sqrt{17}}$, $\cos\theta = -\frac{1}{\sqrt{17}}$, $\cot\theta = -\frac{1}{4}$

d. $\sin\theta = \frac{\sqrt{2}}{10}$ $\cos\theta = \frac{7\sqrt{2}}{10}$, $\tan\theta = \frac{1}{7}$ 22. a. $\frac{3}{25}$ b. $\frac{1}{13}$ c. $\frac{7}{4}$ d. 4 23. a. $\cot\alpha = \frac{1}{3}$ b. $\cos\alpha = -\frac{3\sqrt{17}}{13}$

c. $\sec\alpha = \frac{2}{\sqrt{3}}$ 24. a. $\sqrt{57}$ b. $\frac{3\sqrt{3}}{2}$ 25. $P = 26$, $A = 21\sqrt{3}$ 26. a. 48° 30′ b. 136° 12′ c. 213°45′ d. 313° 47′ 27. a. 121.25°

b. 346.8$\overline{3}$° c. 198.32° d. 23.94° 28. a. 0.3971 b. 0.5842 c. 2.9208 d. 0.4975 29. a. 0.8549 b. 0.9737 c. −1.1323

d. 0.0032

EXERCISES 4

1. a. $\sqrt{13-6\sqrt{3}}$ b. $\frac{-3+\sqrt{37}}{2}$ c. $1+\sqrt{13}$ d. $2\sqrt{5-2\sqrt{3}}$ 2. a. $\frac{\sqrt{113}}{3}$ b. $\sqrt{13}$ c. $\frac{\sqrt{14}}{2}$ d. $\frac{\sqrt{19}}{2}$ e. 2 f. $\frac{\sqrt{79}}{2}$ 3. 120°

4. 135° 5. a. $\sqrt{2}+\sqrt{6}$ b. $\sqrt{2}$ c. 9.6 6. a. $\frac{5}{6}$ b. $\frac{7\sqrt{3}-\sqrt{15}}{16}$ c. $\frac{\sqrt{10}}{10}$ d. $\frac{2}{3}$ 7. $\frac{14\sqrt{3}}{3}\pi$ units 8. a. $5\sqrt{6}$ cm b. $5\sqrt{2}$ cm

9. 6 cm 10. $m(\angle A) = 30°$, $m(\angle B) = 60°$, $m(\angle C) = 90°$ 11. 84 cm^2 12. $\frac{17}{2}$ units 13. a. $\frac{\sqrt{2}}{4}(\sqrt{3}+1)$

b. $\frac{\sqrt{2}}{4}(\sqrt{3}-1)$ c. $2+\sqrt{3}$ d. $\frac{\sqrt{2}}{4}(1-\sqrt{3})$ e. $\sqrt{3}-2$ f. $2-\sqrt{3}$ g. $\frac{\sqrt{2}}{4}(1-\sqrt{3})$ h. $-2-\sqrt{3}$ 14. a. $\frac{1}{6}+\frac{5\sqrt{3}}{12}$

b. $\frac{1}{6}-\frac{5\sqrt{3}}{12}$ c. $\frac{\sqrt{5}}{6}\left(\frac{1}{2}-\sqrt{3}\right)$ d. $\frac{\sqrt{5}}{6}\left(\frac{1}{2}+\sqrt{3}\right)$ e. $-\frac{\sqrt{5}}{55}(9\sqrt{3}+32)$ f. $-\frac{\sqrt{5}}{71}(9\sqrt{3}+32)$ 15. a. 1 b. $\frac{3\sqrt{34}}{34}$

c. $\frac{\sqrt{2}}{2}$ d. $\frac{47\sqrt{17}}{289}$ e. 0 f. $\frac{4\sqrt{17}}{17}$ 16. a. $\cos x$ b. $2\cos x\cos y$ c. $\sqrt{3}\sin x$ d. $2\sin y\cos x$

17. $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$, $\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$ 18. a. $\sin 2x = \frac{24}{25}$, $\cos 2x = \frac{7}{25}$, $\tan 2x = \frac{24}{7}$

b. $\sin 2x = -\frac{120}{169}$, $\cos 2x = -\frac{119}{169}$, $\tan 2x = \frac{120}{119}$ c. $\sin 2x = -\frac{336}{625}$, $\cos 2x = \frac{527}{625}$, $\tan 2x = -\frac{336}{527}$

d. $\sin 2x = -\frac{\sqrt{15}}{8}$, $\cos 2x = \frac{7}{8}$, $\tan 2x = -\frac{\sqrt{15}}{7}$ e. $\sin 2x = \frac{12}{13}$, $\cos 2x = -\frac{5}{13}$, $\tan 2x = -\frac{12}{5}$

19. a. $\sin\frac{x}{2}=\frac{\sqrt{5}}{5}$, $\cos\frac{x}{2}=\frac{2\sqrt{5}}{5}$, $\tan\frac{x}{2}=\frac{1}{2}$ b. $\sin\frac{x}{2}=\frac{\sqrt{5}}{5}$, $\cos\frac{x}{2}=-\frac{2\sqrt{5}}{5}$, $\tan\frac{x}{2}=-\frac{1}{2}$

c. $\sin\frac{x}{2}=\frac{1}{\sqrt{18+12\sqrt{2}}}$, $\cos\frac{x}{2}=\frac{2\sqrt{2}+3}{\sqrt{18+12\sqrt{2}}}$, $\tan\frac{x}{2}=\frac{1}{2\sqrt{2}+3}$

d. $\sin\frac{x}{2}=\frac{1}{\sqrt{4+2\sqrt{2}}}$, $\cos\frac{x}{2}=\frac{\sqrt{2}+1}{\sqrt{4+2\sqrt{2}}}$, $\tan\frac{x}{2}=\sqrt{2}-1$ e. $\sin\frac{x}{2}=\frac{\sqrt{6}}{6}$, $\cos\frac{x}{2}=-\frac{\sqrt{30}}{6}$, $\tan\frac{x}{2}=-\frac{\sqrt{5}}{5}$

f. $\sin\frac{x}{2}=\frac{1}{\sqrt{52+10\sqrt{26}}}$, $\cos\frac{x}{2}=-\frac{5+\sqrt{26}}{\sqrt{52+10\sqrt{26}}}$, $\tan\frac{x}{2}=-\frac{1}{5+\sqrt{26}}$ 20. a. $\frac{\sqrt{2}}{2}$ b. $\frac{\sqrt{6}}{2}$ c. $-\frac{\sqrt{2}}{2}$ d. $\frac{\sqrt{2}}{2}(\sqrt{3}-1)$

e. $\frac{\sqrt{2}}{2}$ f. 0 g. $\frac{\sqrt{2}}{2}$ h. $\sqrt{2}\sin\frac{\pi}{8}$ 21. a. $2\sin 4x\cdot\cos x$ b. $-2\cos\frac{5x}{2}\cdot\sin\frac{3x}{2}$ c. $2\sin 5x\cdot\sin x$ d. $2\cos\frac{11x}{2}\cdot\cos\frac{7x}{2}$

e. $-2\cos\frac{9x}{2}\cdot\sin\frac{5x}{2}$ f. $2\sin\frac{7x}{2}\cdot\cos\frac{x}{2}$ g. $\sin 11x+\sin 9x=2\sin 10x\cdot\cos x$ h. $2\sin\frac{3x}{2}\cdot\sin x$ 24. $\frac{25}{19}$ 25. $-\frac{33}{65}$

26. $-\frac{8}{11}$ 27. $\frac{3\sqrt{10}}{10}$ 28. $\frac{2}{(n+1)^2}$ 29. $\frac{3}{4}$ 30. $-\frac{33}{56}$ 31. $\sin 40°=2x\sqrt{1-x^2}$ 32. $-\frac{55}{24}$ 33. $-\frac{3}{4}$ 34. $2-\sqrt{3}$

35. $-\frac{6}{5}$ 36. $2\sqrt{6}$ 37. $\frac{1}{16}$ 38. $\frac{\sqrt{3}+2x}{4}$ 39. $-\frac{\sqrt{3}}{4}$ 40. $-\frac{\sqrt{2}}{4}$ 41. 1

EXERCISES 5

1. 1381.63 m 2. 2787.69 m 3. 106.90 m 4. 123.21 km 5. 78.32 m 6. a. 134.42° b. 12 m c. 33.80 m^2

7. a. 9.55° b. 9.25 m 8. 45.81° 9. a. $\frac{d\cdot\tan\alpha}{\sin\beta-\cos\beta\cdot\tan\alpha}$ c. 587.73 m 10. 56.04°

12. $h\cong 18.82$ m, $x\cong 83.62$ m 13. 2.04 m 14. 102.61 m/s 15. $h\cong 22.18$ m, $V\cong 925.57$ m^3

ANSWERS TO TESTS

TEST 1

1.	E	9.	E
2.	A	10.	C
3.	B	11.	C
4.	E	12.	D
5.	E	13.	A
6.	E	14.	B
7.	A	15.	D
8.	E	16.	C

TEST 2

1.	D	9.	E
2.	B	10.	B
3.	D	11.	E
4.	B	12.	C
5.	C	13.	A
6.	D	14.	D
7.	D	15.	C
8.	E	16.	D

TEST 3

1.	E	9.	C
2.	B	10.	E
3.	B	11.	E
4.	A	12.	C
5.	C	13.	A
6.	A	14.	C
7.	A	15.	B
8.	D	16.	C

GLOSSARY

A

angle: the union of two rays which have a common endpoint.

angle of elevation (angle of inclination): If an object being observed is above the horizontal, the angle between the line of sight and the horizontal is called the angle of elevation or angle of inclination.

angle of depression (angle of declination): If an object being observed is below the horizontal, the angle between the line of sight and the horizontal is called the angle of depression or angle of declination.

arc: a segment of a circle between the two sides of an angle.

C

circumference: the perimeter of a circle, which measures $2\pi r$ units (r = radius of the circle).

cosecant ratio: In a right triangle, cosec θ is the ratio of the length of the hypotenuse and the length of the side opposite the angle θ. $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$.

cosine ratio: In a right triangle, $\cos\theta$ is the ratio of the length of the adjacent side and the length of the hypotenuse.

cotangent ratio: In a right triangle, $\cot\theta$ is the ratio of the length of the adjacent side and the length of the opposite side. $\cot\theta = \frac{1}{\tan\theta}$.

coterminal angles: two or more angles whose terminal sides coincide with each other when they are in standard position.

coordinate plane: a plane in which the location of a point is given by a pair of coordinates (x, y) that represent horizontal and vertical distances along the x-axis and y-axis, respectively.

chord: a line segment which connects two points on a circle.

cofunction: the trigonometric function of the complement of an angle.

complete angle: the angle which corresponds to one full revolution around a circle. A complete angle measures 360 degrees, 400 grads or 2π radians.

circumscribed circle: a circle around a polygon which passes through all the vertices of the polygon.

D

degree: When the circumference of a circle is divided into 360 equal parts, the central angle corresponding to one of these parts is called 1 degree and is denoted by 1°.

directed angle: an angle which has an initial side and a terminal side.

diameter: a chord which passes through the center of a circle.

directed arc: an arc which has positive or negative direction.

G

grad: When the circumference of a circle is divided into 400 equal parts, the central angle corresponding to one of these arcs is called 1 grad and is denoted by 1^{G}.

M

minute: an angle measure equal to 1/60 of a degree, denoted by ′. 20′ means 20 minutes, i.e. 1/3 of a degree.

N

negative angle: an angle which is measured clockwise from its initial side.

O

origin: the point $O(0, 0)$ in the coordinate plane.

P

positive angle: an angle which is measured counter-clockwise from its initial side.

Q

quadrantal angle: an angle in standard position whose terminal side lies along a coordinate axis.

R

radian: On a circle, one radian is the measure of the central angle corresponding to an arc which has the same length as the radius. It is denoted by 1 rad or 1^R.

reciprocal: the multiplicative inverse of a number or function. Sine and cosecant are reciprocal trigonometric functions since $\sin\theta = \frac{1}{\sec\theta}$.

Similarly, $\cos\theta = \frac{1}{\sec\theta}$ and $\tan\theta = \frac{1}{\cot\theta}$.

reduction formulas: a set of formulas which can be used to reduce a trigonometric function value in any quadrant to a trigonometric value in the first quadrant.

reference angle: the positive acute angle α which is formed by the terminal side of a nonquadrantal angle θ and the x-axis.

S

second: an angle measure equal to 1/60 of a minute (i.e. 1/3600 of a degree), denoted by ″. 20″ means 20 seconds, i.e. 2/360 of a degree.

secant ratio: In a right triangle, $\sec\theta$ is the ratio of the hypotenuse to the adjacent side. $\sec\theta = \frac{1}{\cos\theta}$.

sine ratio: In a right triangle, $\sin\theta$ is the ratio of the opposite side to the hypotenuse.

standard equation of a circle: the equation $(x-a)^2 + (y-b)^2 = r^2$ which defines a circle with radius r and center $C(a, b)$.

standard position of an angle: An angle in the coordinate plane whose vertex is at the origin and whose initial side lies along the positive x-axis is said to be in standard position.

secant line: a line which passes through at least two points on a circle.

similar triangles: Two or more triangles are similar if their corresponding angles are equal and their corresponding sides are in the same proportion.

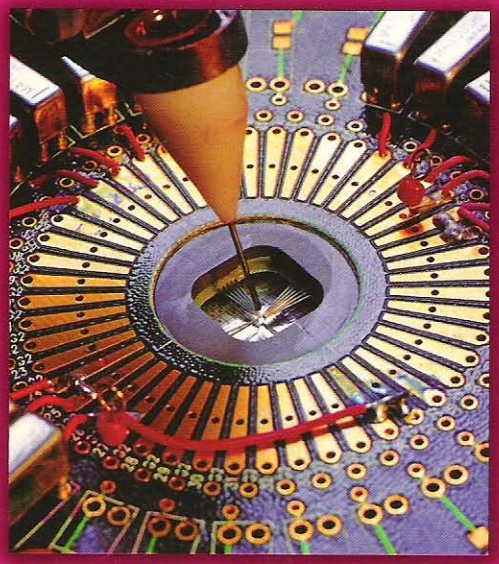
T

tangent ratio: In a right triangle, $\tan\theta$ is the ratio of the opposite side to the adjacent side.

triangulation: the process of dividing a polygon in a plane into a set of triangles, usually with the restriction that each triangle side is entirely shared by two adjacent triangles.

U

unit circle: a circle in the coordinate plane which has its center at the origin (0, 0) and radius 1 unit.



Introduction to TRIGONOMETRY

MODULAR SYSTEM

Zambak PUBLISHING

ISBN 975-266-471-7

9 789752 664715